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THE UNCONDITIONAL CONVERGENT DIFFERENCE METHODS WITH INTRINSIC PARALLELISM FOR QUASILINEAR PARABOLIC SYSTEMS WITH TWO DIMENSIONS*1)

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Dedicated to the 80th birthday of Professor Zhou Yulin

Abstract

In the present work we are going to solve the boundary value problem for the quasilinear parabolic systems of partial differential equations with two space dimensions by the finite difference method with intrinsic parallelism. Some fundamental behaviors of general finite difference schemes with intrinsic parallelism for the mentioned problems are studied. By the method of a priori estimation of the discrete solutions of the nonlinear difference systems, and the interpolation formulas of the various norms of the discrete functions and the fixed–point technique in finite dimensional Euclidean space, the existence of the discrete vector solutions of the nonlinear difference system with intrinsic parallelism are proved. Moreover the convergence of the discrete vector solutions of these difference schemes to the unique generalized solution of the original quasilinear parabolic problem is proved.

Key words: Difference Scheme, Intrinsic Parallelism, Two Dimensional Quasilinear Parabolic System, Existence, Convergence.

1. Introduction

1. In [1]-[4] the finite difference methods with intrinsic parallelism for the multi-dimensional boundary value problems of the semilinear parabolic system are studied, where the difference approximations for the derivatives of second order are taken to be the various linear combinations of the two or four kinds of difference quotients. All of these general finite difference schemes having the intrinsic character of parallelism are proved to be stable and convergent conditionally, where some restriction conditions on time–step must be satisfied. Some special finite difference schemes with intrinsic parallelism for the linear parabolic problems have been discussed in [5] and [6]. These special difference schemes are proved to be stable and convergent unconditionally in discrete norms L^∞ and H^1, and the convergence order is $O(\tau + h)$ though the truncation error at the subdomain boundaries is $O(1)$. For the one-dimensional quasilinear parabolic systems we have also constructed some general difference schemes with intrinsic parallelism and proved that they are unconditional stable and convergent in [7].

2. Difference Schemes with Intrinsic Parallelism

2. Consider the boundary value problems for the two dimensional quasilinear parabolic systems of second order of the form

$$u_t = A(x,y,t,u)(u_{xx} + u_{yy}) + B(x,y,t,u)u_x + C(x,y,t,u)u_y + f(x,y,t,u) \qquad (1)$$

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where $(x,y)\in\Omega=(0,l_1)\times(0,l_2)$, $t\in(0,T]$, and $u(x,y,t)=(u_1(x,y,t),u_2(x,y,t),\cdots,u_m(x,y,t))$ is a m–dimensional vector unknown function $(m\geq 1)$; $A(x,y,t,u)$, $B(x,y,t,u)$ and $C(x,y,t,u)$ are given $m\times m$ matrix functions, and $f(x,t,u)$ is a given m–dimensional vector function and $u_x=\frac{\partial u}{\partial x}$, $u_y=\frac{\partial u}{\partial y}$, $u_{xx}=\frac{\partial^2 u}{\partial x^2}$, $u_{yy}=\frac{\partial^2 u}{\partial y^2}$ and $u_t=\frac{\partial u}{\partial t}$ are the corresponding m–dimensional vector derivatives of the m–dimensional unknown vector function $u(x,y,t)$.

Let us consider in the rectangular domain $Q_T=\bar{\Omega}\times[0,T]$ the boundary value problem for the system (1) with homogeneous boundary conditions

$$u(x,y,t)=0,\quad (x,y)\in\partial\Omega, 0<t\leq T, \tag{2}$$

and the initial condition

$$u(x,y,0)=\varphi(x,y),\quad (x,y)\in\bar{\Omega}. \tag{3}$$

Suppose that the following conditions are satisfied.

(I) $A(x,y,t,u)$, $B(x,y,t,u)$, $C(x,y,t,u)$and $f(x,y,t,u)$ are continuous functions with respect to $(x,y,t)\in Q_T$ and continuously differentiable with respect to $u\in R^m$; and there are constants $A_0>0$, $B_0>0, C_0>0$ and $C>0$ such that $|A(x,y,t,u)|\leq A_0$, $|B(x,y,t,u)|\leq B_0, |C(x,y,t,u)|\leq C_0$, and $|f(x,y,t,u)|\leq|f(x,y,t,0)|+C|u|$.

(II) There is a constant $\sigma_0>0$, such that, for any vector $\xi\in R^m$, and for $(x,y,t,u)\in Q_T\times R^m$,

$$(\xi,A(x,y,t,u)\xi)\geq\sigma_0|\xi|^2.$$

(III) The initial value m–dimensional vector function $\varphi(x,y)\in C^1(\bar{\Omega})$ and $\varphi(x,y)=0$ for $(x,y)\in\partial\Omega$.

3. Divide the domain $Q_T=\{0\leq x\leq l_1, 0\leq y\leq l_2, 0\leq t\leq T\}$ into small grids by the parallel planes $x=x_i$ $(i=0,1,\cdots,I)$, $y=y_j$ $(j=0,1,\cdots,J)$ and $t=t^n$ $(n=0,1,\cdots,N)$ with $x_i=ih_1$, $y_j=jh_2$ and $t^n=n\tau$, where $Ih_1=l_1, Jh_2=l_2$ and $N\tau=T$, I,J and N are integers and h_1,h_2 and τ are the steplengths of grids. Denote $v_\Delta=\{v_{ij}^n|i=0,1,\cdots,I;j=0,1,\cdots,J;n=0,1,\cdots,N\}$ the m–dimensional discrete vector function defined on the discrete rectangular domain $Q_\Delta=\{(x_i,y_j,t^n)|i=0,1,\cdots,I;j=0,1,\cdots,J;n=0,1,\cdots,N\}$ of the grid points.

Let us now construct the general difference schemes with intrinsic parallelism for the boundary value problem (1), (2) and (3):

$$\frac{v_{ij}^{n+1}-v_{ij}^n}{\tau}=A_{ij}^{n+1}\overset{*}{\Delta}v_{ij}^{n+1}+B_{ij}^{n+1}\bar{\delta}_x^1 v_{ij}^{n+1}+C_{ij}^{n+1}\bar{\delta}_y^1 v_{ij}^{n+1}+f_{ij}^{n+1}, \tag*{$(1)_\Delta$}$$
$$(i=1,2,\cdots,I-1;j=1,2,\cdots,J-1;n=0,1,\cdots,N-1),$$

where

$$\begin{aligned}\overset{*}{\Delta}v_{ij}^{n+1}&=\overset{*}{\delta}{}_x^2 v_{ij}^{n+1}+\overset{*}{\delta}{}_y^2 v_{ij}^{n+1}\\&=\frac{v_{i+1,j}^{n+\lambda_{ij}^n}-2v_{ij}^{n+1}+v_{i-1,j}^{n+\mu_{ij}^n}}{h_1^2}+\frac{v_{i,j+1}^{n+\bar{\lambda}_{ij}^n}-2v_{ij}^{n+1}+v_{i,j-1}^{n+\bar{\mu}_{ij}^n}}{h_2^2},\\A_{ij}^{n+1}&=A(x_i,y_j,t^{n+1},\bar{\delta}^0 v_{ij}^{n+1}),\\B_{ij}^{n+1}&=B(x_i,y_j,t^{n+1},\tilde{\delta}^0 v_{ij}^{n+1}),\\C_{ij}^{n+1}&=C(x_i,y_j,t^{n+1},\hat{\delta}^0 v_{ij}^{n+1}),\\f_{ij}^{n+1}&=f(x_i,y_j,t^{n+1},\check{\delta}^0 v_{ij}^{n+1}).\end{aligned}\tag{4}$$

In this difference scheme, the expressions $\bar{\delta}^0 v_{ij}^{n+1}$, $\tilde{\delta}^0 v_{ij}^{n+1}$, $\hat{\delta}^0 v_{ij}^{n+1}$, $\check{\delta}^0 v_{ij}^{n+1}$, and $\bar{\delta}_x^1 v_{ij}^{n+1}, \bar{\delta}_y^1 v_{ij}^{n+1}$ can be taken in the following manner. We can take

$$\begin{aligned}\bar{\delta}^0 v_{ij}^{n+1}\;&=\;\lambda_{ij}^n\alpha_{1ij}^n v_{i+1j}^{n+1}+\mu_{ij}^n\alpha_{2ij}^n v_{i-1j}^{n+1}+\bar{\lambda}_{ij}^n\alpha_{3ij}^n v_{ij+1}^{n+1}\\&+\bar{\mu}_{ij}^n\alpha_{4ij}^n v_{ij-1}^{n+1}+\alpha_{5ij}^n v_{ij}^{n+1}+\bar{\alpha}_{1ij}^n v_{i+1j}^n\\&+\bar{\alpha}_{2ij}^n v_{i-1j}^n+\bar{\alpha}_{3ij}^n v_{ij+1}^n+\bar{\alpha}_{4ij}^n v_{ij-1}^n+\bar{\alpha}_{5ij}^n v_{ij}^n\end{aligned}\tag{5}$$