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FINITE ELEMENT METHODS FOR SOBOLEV EQUATIONS*[1)]

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Abstract

A new high-order time-stepping finite element method based upon the high-order numerical integration formula is formulated for Sobolev equations, whose computations consist of an iteration procedure coupled with a system of two elliptic equations. The optimal and superconvergence error estimates for this new method are derived both in space and in time. Also, a class of new error estimates of convergence and superconvergence for the time-continuous finite element method is demonstrated in which there are no time derivatives of the exact solution involved, such that these estimates can be bounded by the norms of the known data. Moreover, some useful a-posteriori error estimators are given on the basis of the superconvergence estimates.

Key words: Error estimates, finite element, Sobolev equation, numerical integration.

1. Introduction

Our purpose in this paper is to study the finite element method for the following Sobolev equation:

$$\begin{aligned} A(t)u_t + \quad B(t)u &= f(t), && \text{in} \quad \Omega \times J, \\ u(\cdot,t) &= 0, && \text{on} \quad \partial\Omega \times \bar{J}, \\ u(\cdot,0) &= v, && x \in \Omega, \end{aligned} \tag{1.1}$$

where $\Omega \subset R^d$ $(d \geq 1)$ is an open bounded domain, $J = (0,T]$, $T > 0$, f and v are known smooth functions. We assume that the operator $A(t)$ is a strongly elliptic symmetric operator,

$$A(t) := -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial}{\partial x_j}\right) + a(x,t)I, \quad a(x,t) \geq 0,$$

and that $B(t)$ is an arbitrary second order elliptic operator,

$$B(t) := -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(b_{ij}(x,t)\frac{\partial}{\partial x_j}\right) + \sum_{i=1}^{d} b_i(x,t)\frac{\partial}{\partial x_i} + b(x,t)I,$$

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where I is the identity operator, a_{ij}, a, b_{ij}, b_i and b are smooth functions, and there exits $C_0 > 0$ such that

$$\sum_{i,j=1}^{d} a_{ij}\xi_i\xi_j \geq C_0 \sum_{i=1}^{d} \xi_i^2, \qquad \forall \xi \in R^d, \quad (x,t) \in \Omega \times \bar{J}. \tag{1.1}$$

The problem (1.1) can arise from many physical processes. For the fomulation of (1.1) and the questions of existence, uniqueness and stability of the solution, we refer to [2, 3, 19] and the references cited in [6, 7, 8]. The numerical approximations to the solution of (1.1) have been investigated by many authors. Finite difference methods have been studied in [6, 10, 11], while Ewing [8] has considered several Galerkin approximations and obtained optimal error estimates for nonlinear boundary cases. Also, Arnold, Douglas and Thomée [1] and Nakao [17] have studied Galerkin approximations to the solution of (1.1) in a single space dimension with periodic boundary conditions. L^2 error estimates and superconvergence results are derived by these authors. Recently, the authors in [14, 15, 16] have used a so-called Ritz-Volterra type projection to study finite element approximations for nonlinear versions of the above problems and derived some optimal error estimates for Dirichlet and nonlinear boundary conditions. The L^p $(2 \leq p < \infty)$ norm error estimate can be found in [16] for linear equations.

In this paper we reformulate (1.1) as an integral equation of Volterra type, use the higher-order numerical integration formula to construct a higher-order time-stepping procedure and give some error estimates. The formulation of our numerical approximations is given in Section 2, and error estimates of convergence and superconvergence for the semi-discrete and the fully-discrete finite element methods are demonstrated in Sections 3, 4 and 5, respectively. The special feature of our error estimates in Sections 3 and 4 compared with the others [1, 6, 7, 8, 12-17] is that there are no time derivatives of the exact solution u of (1.1) involved in the analysis and the results, such that these estimates are bounded by the norms of the known data v and f.

2. Formulation of finite element methods

Let S_h be a family of finite element subspaces of $H_0^1(\Omega)$ with the following standard approximation properties: For some $l \geq 1$,

$$\inf_{\chi \in S_h} (||\chi - w|| + h||\chi - w||_1) \leq Ch^{r+1}||w||_{r+1}, \quad 1 \leq r \leq l, \quad w \in H^{r+1}(\Omega) \cap H_0^1(\Omega), \tag{2.1}$$

where $C > 0$ is a constant independent of h, and $||\cdot||_m$ is the norm in the Hilbert space $H^m(\Omega)$ with $||\cdot|| = ||\cdot||_0$, and $H_0^1(\Omega)$ is the completion of $C_0^\infty(\Omega)$ under the norm $||\cdot||_1$.

The time-continuous finite element approximation to the solution u of (1.1) can now be defined as a mapping $u_h(t) : \bar{J} \to S_h$ by

$$\begin{aligned} A(t; u_{h,t}, \chi) + \quad B(t; u_h, \chi) &= (f, \chi), \quad \chi \in S_h, \\ u_h(0) &= v_h \end{aligned} \tag{2.2}$$

where v_h is an appropriate approximation of v into S_h, $A(t;\cdot,\cdot)$ and $B(t;\cdot,\cdot)$ are the bilinear forms associated with the operators $A(t)$ and $B(t)$ on $H_0^1(\Omega) \times H_0^1(\Omega)$.

Before we define the fully-discrete method, let us define (see, for example, [16]) $A_h(t)$: $S_h \to S_h$ by

$$(A_h(t)\phi, \psi) = A(t; \phi, \psi), \quad \forall \phi, \psi \in S_h \tag{2.3}$$

and $B_h(t) : S_h \to S_h$ by

$$(B_h(t)\phi, \psi) = B(t; \phi, \psi), \quad \forall \phi, \psi \in S_h. \tag{2.4}$$

Also, we define the L^2-projection operator $P_h : L^2(\Omega) \to S_h$, for any $w \in L^2(\Omega)$, by

$$(P_h w - w, \chi) = 0, \quad \forall \chi \in S_h. \tag{2.5}$$