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ON THE *hp* FINITE ELEMENT METHOD FOR THE ONE DIMENSIONAL SINGULARLY PERTURBED CONVECTION-DIFFUSION PROBLEMS*[1)]

Zhi-min Zhang
(*Department of Mathematics, Wayne State University, Detroit, MI 48202, U.S.A.*)

Abstract

In this work, a singularly perturbed two-point boundary value problem of convection-diffusion type is considered. An *hp* version finite element method on a strongly graded piecewise uniform mesh of Shishkin type is used to solve the model problem. With the analytic assumption of the input data, it is shown that the method converges exponentially and the convergence is uniformly valid with respect to the singular perturbation parameter.

Key words: *hp*-version finite element methods, convection-diffusion, singularly perturbed, exponential rate of convergence.

1. Introduction

In practice, we often encounter differential equations with small (or large) parameters. When these parameters go to extremal, the equations are usually *singularly perturbed*. One typical behavior of the singular perturbation is the so-called *boundary layers*. The existence of the boundary layers causes difficulty in numerically solving these problems. The conventional methods fail to converge since the convergence deteriorated at the limits of the small (large) parameters. A successful numerical algorithm should converge uniformly with respect to singular perturbation parameters. There is a rich literature on numerical methods for problems with boundary layers. The reader is referred to recent books of Miller et al. [13], Morton [14], Roos et al. [15], and references therein.

Concerning singularly perturbed problems, it is common knowledge that solving convection-diffusion equations is usually harder than solving reaction-diffusion equations. The main difficulty with generalizing the theoretical analysis for reaction-diffusion problems to convection-diffusion problems is that the bilinear forms of the latter are not uniformly continuous with respect to the singular perturbation parameter ϵ. To be more precise, for the reaction-diffusion problem, there exists a constant C independent of ϵ, so that the inequality

$$|B_\epsilon(u,v)| \le C\|u\|_\epsilon \|v\|_\epsilon \tag{1.1}$$

holds for an energy norm $\|\cdot\|_\epsilon$. Here $B_\epsilon(\cdot,\cdot)$ is the bilinear form of the variational formulation. However, this property is not valid for the convection-diffusion problem. The lack of the stability property (1.1) prevents us from following the standard analysis. In order to overcome this difficulty, many methods are suggested in the literature, among which, the most popular are streamline-diffusion technique and the Petrov-Galerkin method (see, e.g., [6, 7, 8, 10, 15]). However, the Petrov-Galerkin method is difficult to be generalized to multidimensional settings and the streamline diffusion method alone is not able to resolve the boundary layer. If the

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boundary layer is concerned, a practical procedure would be to start with the streamline diffusion method, adapting the mesh by an *a posteriori* error estimate, and eventually resolving the boundary layer. An important question naturally arises: what is the quality of the numerical approximation when the mesh is consistent with the boundary layer?

In this work, a singularly perturbed two-point boundary value problem of convection-diffusion type is considered. A special hp finite element method that uses piecewise uniform meshes, a uniform mesh outside the boundary layer region and a much smaller uniform mesh in the boundary layer region, is applied. The convergence analysis avoids the use of (1.1) by adopting a different framework from the traditional one. Furthermore, the analysis is carried out on the element level which allows the use of some fundamental results form approximation theory to tracking the exact dependence on p and h, thereby to compensate the boundary layer influence. The main result of this paper is to establish, under the analytic assumption of the input data, an exponential convergent rate for the energy norm, a rate which is uniformly valid with respect to the singular perturbation parameter ϵ for the proposed method.

In an independent work done recently by Melenk and Schwab [12], the authors obtained exponential convergence for both hp and hp streamline diffusion finite element methods. However, the current approach is simpler which provides explicit dependence of the convergent rate on the regularity constants. Furthermore, the proof here is elementary and self-contained.

Since the publication of the first theoretical paper [2] on the p-version finite element method, many works have been done on the p and hp methods. For the general information, the reader is referred to [1, 5, 16, 18] and references therein.

2. Main Results

Consider the following steady state one-dimensional convection-diffusion model problem.

$$(L_\epsilon u)(x) = -\epsilon u''(x) + a(x)u'(x) + b(x)u(x) = f(x) \quad \text{in} \quad \gamma = (0,1), \quad u(0) = u(1) = 0 \tag{2.1}$$

with

$$a(x) \geq \alpha > 0, \quad b(x) - \frac{a'(x)}{2} > 0 \quad \forall x \in \bar{\gamma}. \tag{2.2}$$

It has been shown in [13, Chapter 9] that there is no essential loss of generality in assuming the above rather than

$$a(x) > \alpha > 0, \quad b(x) \geq \beta, \quad \forall x \in \bar{\gamma}. \tag{2.3}$$

Numerical difficulty arises when the diffusion parameter ϵ is small. In this case, the model problem is singularly perturbed. In order to design a good numerical algorithm, it is necessary to understand the boundary layer behavior of the problem. This understanding involves regularity analysis based on the input data. In this work, we utilize the regularity result in [10] for (2.1) under the analytic assumption on the input data. For small ϵ, the solution u can be decomposed into

$$u = \sum_{j=0}^{m} \epsilon^j u_j + u_\epsilon + r_m = w_m + u_\epsilon + r_m \tag{2.4}$$

where u_j, u_ϵ, and r_m are determined by the following initial value problems and boundary value problems:

$$\begin{aligned}
a(x)u_0'(x) + b(x)u_0(x) &= f(x), \quad u_0(0) = 0; \\
a(x)u_{j+1}'(x) + b(x)u_{j+1}(x) &= u_j''(x), \quad u_{j+1}(0) = 0, \quad j = 0, 1, \ldots, m-1; \\
L_\epsilon u_\epsilon &= 0, \quad u_\epsilon(0) = 0, \quad u_\epsilon(1) = -w_m(1); \\
L_\epsilon r_m &= \epsilon^{m+1} u_m'', \quad r_m(0) = 0 = r_m(1).
\end{aligned}$$