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# NONLINEAR STABILITY OF NATURAL RUNGE-KUTTA METHODS FOR NEUTRAL DELAY DIFFERENTIAL EQUATIONS[*1)]

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**Abstract**

This paper first presents the stability analysis of theoretical solutions for a class of nonlinear neutral delay-differential equations (NDDEs). Then the numerical analogous results, of the natural Runge-Kutta (NRK) methods for the same class of nonlinear NDDEs, are given. In particular, it is shown that the $(k,l)$-algebraic stability of an RK method for ODEs implies the generalized asymptotic stability and global stability of the induced NRK method.

*Key words*: Nonlinear stability, Neutral delay differential equations, Natural Runge-Kutta methods.

## 1. Introduction

In the last several decades, there has been a growing interest in the numerical stability for DDEs(cf. [1-14]). In 1988, A.Bellen, Z. Jackiewicz and M.Zennaro[7] first extend the researches to the scalar linear NDDEs. Latterly, a lot of works for the systems of linear NDDEs were presented(cf.[8-12]). However, there are much difficulties to assess the numerical stability of nonlinear NDDEs. In view of this, T.Koto [13] adapted NRK methods (cf.[14]) to a class of nonlinear NDDEs in real space $\mathbf{R}^d$, and studied their asymptotic stability with a discrete analogue of the Liapunov functional.

In this paper, by an alternative approach, we futher deal with the stability of theoretical and numerical solutions for a class of nonlinear NDDEs in complex space $\mathbf{C}^d$. Particularly, it is shown that a NRK method induced by a $(k,l)-$algebraically stable RK methods for ODEs, under suitable conditions,preserves the analogous stability of the original equations.

## 2. Test Problem and Its Stability

For giving subsequent analysis, we first set some notational conventions. Let $\langle\bullet,\bullet\rangle, \|\bullet\|$ denote the inner product and the induced norm in space $\mathbf{C}^d$, respectively. Correspondingly, the inner product and the induced norm in space $(\mathbf{C}^d)^l$ are defined as follows:

$$\langle U,V\rangle=\sum_{i=1}^{l}\langle u_i,v_i\rangle,\qquad \|U\|^2=\langle U,U\rangle,$$

where $U=(u_1,u_2,\dots,u_l), U=(v_1,v_2,\dots,v_l)\in(\mathbf{C}^d)^l$ and $u_i,v_i\in\mathbf{C}^d(i=1,2,\dots,l)$. Moreover, it is always assumed that each matrix norms, arising in the following, is subject to the corresponding vector norm.

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Consider the following systems of nonlinear NDDEs

$$\begin{cases} \frac{d}{dt}[y(t) - Ny(t-\tau)] = f(t, y(t), y(t-\tau)), \quad t \geq 0, \\ y(t) = \phi(t), \quad -\tau \leq t \leq 0, \end{cases} \tag{2.1}$$

and

$$\begin{cases} \frac{d}{dt}[z(t) - Nz(t-\tau)] = f(t, z(t), z(t-\tau)), t \geq 0, \\ z(t) = \psi(t), -\tau \leq t \leq 0, \end{cases} \tag{2.2}$$

where $\tau > 0$ is constant delay, $N \in \mathbf{C}^{d\times d}$ stand for a constant matrix with $\|N\| < 1$, $\phi, \psi: [-\tau, 0] \to \mathbf{C}^d$ are continuous functions, and $f: [0, +\infty) \times \mathbf{C}^d \times \mathbf{C}^d \to \mathbf{C}^d$ is a assigned mapping subject to

$$\begin{aligned} & \mathrm{Re}\langle (x_1 - x_2) - N(y_1 - y_2), f(t, x_1, y_1) - f(t, x_2, y_2)\rangle \\ \leq \; & \alpha\|x_1 - x_2\|^2 + \beta\|y_1 - y_2\|^2, \quad t \geq 0, \;\; x_1, x_2, y_1, y_2 \in \mathbf{C}^d, \end{aligned} \tag{2.3}$$

in which $\alpha, \beta$ are real constants.

The problems of the form (2.1) can be found in the systems with lossless transmission lines (cf.[16]). In the following,all the problems (2.1) with (2.3) will be referred as *the class* $R_{\alpha,\beta}$. For instance, a complex d-dimensional linear system

$$\begin{cases} \frac{d}{dt}[y(t) - Ny(t-\tau)] = Ly(t) + My(t-\tau), \quad t \geq 0, \\ y(t) = \phi(t), \quad -\tau \leq t \leq 0, \end{cases}$$

belongs to the class $R_{\alpha,\beta}$ whenever the matrix

$$G = \frac{1}{2}\begin{pmatrix} L + L^* - 2\alpha I & M - L^*N \\ M^* - N^*L & -N^*M - M^*N - 2\beta I \end{pmatrix}$$

is negative definite, where $I$ denote a d-dimensional identity matrix and $*$ is the conjugate transpose symbol of the matrices, since

$$\begin{aligned} & Re\langle (x_1 - x_2) - N(y_1 - y_2), L(x_1 - x_2) + M(y_1 - y_2)\rangle \\ & -\alpha\|x_1 - x_2\|^2 - \beta\|y_1 - y_2\|^2 \\ = \; & \langle \begin{pmatrix} x_1 - x_2 \\ y_1 - y_2 \end{pmatrix}, G\begin{pmatrix} x_1 - x_2 \\ y_1 - y_2 \end{pmatrix}\rangle, \quad \forall x_1, x_2, y_1, y_2 \in \mathbf{C}^d. \end{aligned}$$

For the problems of the class $R_{\alpha,\beta}$, we obtain the following stability results.

**Theorem 2.1.** *Suppose problems (2,1),(2.2) belong to the class* $R_{\alpha,\beta}$ *with*

$$\alpha \leq 0, \quad \beta \leq \alpha\|N\|^2. \tag{2.4}$$

*Then we have*

(a) $\|y(t) - z(t)\| \leq \frac{2}{1-\|N\|} \max\limits_{-\tau\leq\theta\leq 0} \|\phi(\theta) - \psi(\theta)\|, \;\; t \geq 0,$

(b) $for \;\; \alpha < 0, \;\; \lim\limits_{t\to+\infty} \|y(t) - z(t) - N(y(t-\tau) - z(t-\tau))\| = 0.$

*Proof.* Let

$$u(t) = y(t) - z(t), \quad v(t) = \|u(t) - Nu(t-\tau)\|^2,$$
$$F(t) = f(t, y(t), y(t-\tau)) - f(t, z(t), z(t-\tau)).$$

Then by (2.3)

$$\begin{aligned} v'(t) \;\; & = 2Re\langle u(t) - Nu(t-\tau), F(t)\rangle \\ & \leq 2[\alpha\|u(t)\|^2 + \beta\|u(t-\tau)\|^2], \quad t \geq 0, \end{aligned}$$