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SPECTRAL ANALYSIS OF THE FIRST-ORDER HERMITE CUBIC SPLINE COLLOCATION DIFFERENTIATION MATRICES*[1)]

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Abstract

It has been observed numerically in [1] that, under certain conditions, all eigenvalues of the first-order Hermite cubic spline collocation differentiation matrices with unsymmetrical collocation points lie in one of the half complex planes. In this paper, we provide a theoretical proof for this spectral result.

Key words: Spline collocation, Differentiation matrices, Spectral analysis.

1. Introduction

Hermite cubic spline collocation method has been extensively applied in the numerical solution of ODEs and PDEs due to its ease of implementation and high-order accuracy[2−5]. One of the main features of this method is that the approximate solution takes the form of Hermite cubic spline(or bicubic spline in 2D). When Hermite cubic spline is applied for discretizing the differential operator $u^{(k)}$, the so-called Hermite cubic spline collocation differentiation matrices arise. Eigenvalue analysis of these matrices plays an important part in the stability or convergence analysis of the corresponding collocation algorithms. It also has served as the theoretical foundation in developing several fast direct algorithms[6−9].

Spectral analysis of second-order Hermite cubic spline collocation differentiation matrices with Gauss collocation points (a kind of symmetric collocation points) was given analytically in [6] and consequently in [7,8] for eigenvectors. However, it has been found that Hermite cubic spline collocation differentiation matrices with unsymmetrical collocation points are more important in practice for stability and singularity considerations[5,11−15]. Based on the condensation technique[16], [1] studied the spectral properties of the second-order Hermite cubic spline collocation differentiation matrices with arbitrary collocation points. [1] also studied the spectral properties of the first-order Hermite cubic spline collocation differentiation matrices with symmetric collocation points. Meanwhile, for unsymmetrical collocation points, [1] observed numerically that all eigenvalues of the first-order Hermite cubic spline collocation differentiation matrices with Dirichlet boundary conditions or Neumann boundary conditions lie in one of the half complex planes. This spectral result is of most importance for stability considerations[1].

The primary purpose of this paper is to provide a theoretical proof for the aforementioned spectral result of the first-order Hermite cubic spline collocation differentiation matrices. Our argumentation is based on the condensation technique and Hurwitz's theorem. The rest of this paper is arranged as follows. In the second section, we will briefly introduce the Hermite cubic spline collocation method and generalize the aforementioned spectral result in two theorems. In the third section, we will prove these theorems theoretically.

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2. Hermite Cubic Spline Collocation Method

Assume that $\Pi_N = \{x_i\}_{i=1}^{N+1}$ is a uniform partition that divides interval $[0,\ 1]$ into N equal subintervals of length $h = 1/N$. The Hermite cubic approximation $v(x)$ is defined by

$$v(x) = \xi_1(s)u_i + h\xi_2(s)u_i' + \xi_3(s)u_{i+1} + h\xi_4(s)u_{i+1}', \quad s \in [0,\ 1], \tag{1}$$

on each subinterval $[x_i,\ x_{i+1}],\ \ i = 1,\ 2,\ \cdots,\ N$, where

$$\xi_1(s) = (1+2s)(1-s)^2,\ \xi_2(s) = s(1-s)^2,\ \xi_3(s) = s^2(3-2s),\ \xi_4(s) = s^2(s-1) \tag{2}$$

and $s = (x - x_i)/h$. The above definition implies that $u_i = v(x_i)$ and $u_i' = v'(x_i)$. Denote

$$\Omega = \{(x,\ y) |\, 0 < x < y < 1\} \subset R^2. \tag{3}$$

For any given $(\sigma_1,\ \sigma_2) \in \Omega$, let $\Pi_C = \{x_{i1}^c,\ x_{i2}^c\}_{i=1}^N$ be the set of collocation points, where

$$x_{i1}^c = x_i + \sigma_1 h,\ x_{i2}^c = x_i + \sigma_2 h,\ i = 1,\ 2,\ \cdots,\ N. \tag{4}$$

Consider a collocation approximation to the kth-order differentiation operator $u^{(k)} (k = 0,\ 1, 2)$. Let $A_k (k = 0,\ 1,\ 2)$ denote the k-order collocation differentiation matrices, satisfying

$$(A_k v)_l = v^{(k)}(x_{ij}^c), \quad j = 1,\ 2; \quad i = 1,\ 2,\ \cdots,\ N. \tag{5}$$

Obviously the structure of A_k depends on the orderings of the collocation points and the unknowns for $v(x)$. Suppose $l = 2(i-1) + j$, and

$$\begin{aligned} &v = [\, hu_1',\ u_2,\ hu_2',\ \cdots,\ u_N,\ hu_N',\ hu_{N+1}']^T \text{ or } [u_1,\ u_2,\ hu_2',\ \cdots,\ u_N,\ hu_N',\ u_{N+1}]^T, \\ &\Pi_C = \{x_{11}^c,\ x_{12}^c, x_{21}^c,\ x_{22}^c,\ \cdots,\ x_{N1}^c,\ x_{N2}^c\}. \end{aligned} \tag{6}$$

Then A_k with Dirichlet boundary conditions or Neumann boundary conditions has an almost block diagonal structure[6,3,1] and A_0 is nonsingular[1]. In this paper we only consider A_k with Dirichlet boundary conditions or Neumann boundary conditions, the corresponding spectral results for A_k with periodical boundary conditions can be found in [10].

Consider the following typical example[13,1]

$$\frac{\partial u}{\partial t} = \varepsilon \frac{\partial^2 u}{\partial x^2} - p\frac{\partial u}{\partial x} + d(x,\ t). \tag{7}$$

If we use an Euler scheme for time discretization and the Hermite cubic collocation method in the x-direction, we shall get the following iterative scheme

$$\begin{cases} u^{n+1} = u^n + \tau(\varepsilon A_2 - pA_1)u^{(n)} + \tau\, d^n, \\ A_0 u^{(n)} = u^n, \end{cases} \tag{8}$$

where τ is the time stepsize. The stability depends on the eigenvalues distribution of the matrix $(\varepsilon\, A_2 - pA_1)A_0^{-1}$. On the one hand it is known that all eigenvlues of $A_2A_0^{-1}$ are real and nonpositive for any collocation points (see [1, 2, 6, 7, 8]). On the other hand, numerical tests show that the stability of (8) depends on the choice of collocation points when $0 < \varepsilon << p$ (see [1]). This indicates that $A_1A_0^{-1}$ plays a dominating part in the case of $0 < \varepsilon << p$. Furthermore, $A_1A_0^{-1}$ is the only factor to determine the stability of (8) when $\varepsilon = 0$. Thus it is very interesting and important to study the spectral properties of matrix $A_1A_0^{-1}$. In this respect, [1] presented a number of concrete formulae for the calculation of all eigenvalues of matrix $A_1A_0^{-1}$ with symmetric collocation points. As for $A_1A_0^{-1}$ with unsymmetrical collocation points, [1] observed