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EXPANSION OF STEP-TRANSITION OPERATOR OF MULTI-STEP METHOD AND ITS APPLICATIONS (II)*1)

Yi-fa Tang

(*State Key Laboratory of Scientific and Engineering Computing, ICMSEC, Academy of Mathematics and System Sciences, Chinese Academy of Sciences, P.O. Box 2719, Beijing 100080, China*)

Abstract

We give some formulae for calculation of the expansions for (1) composition of step-transition operators (STO) of any two difference schemes (DS) for ODE's, (2) inverse operator of STO of any DS, and (3) conjugate operator of STO of any DS.

Key words: Step-transition operator, Expansion, Composition, Inverse operator, Conjugate operator.

1. Introduction

For an ordinarily differential equation (ODE)

$$\frac{d}{dt}Z = f(Z), \quad Z \in R^p, \tag{1}$$

any compatible linear m-step difference scheme (DS)

$$\sum_{k=0}^{m} \alpha_k Z_k = \tau \sum_{k=0}^{m} \beta_k f(Z_k) \quad \left(\sum_{k=0}^{m} \beta_k \neq 0\right), \tag{2}$$

can be characterized by a step-transition operator (STO) G (also denoted by G^τ): $R^p \to R^p$ satisfying

$$\sum_{k=0}^{m} \alpha_k G^k = \tau \sum_{k=0}^{m} \beta_k f \circ G^k, \tag{3}$$

where G^k stands for k-time composition of G: $G \circ G \cdots \circ G$ (refer to [2,5,6,10,11]). This operator G^τ can be represented as a power series in τ with first term equal to *identity* I. More precisely, one can expand[13] the STO $G^\tau(Z)$ of any linear multi-step method (LMSM)2) of form (2) with order $s \geq 2$ up to $O(\tau^{s+5})$:

$$G^\tau(Z) = \sum_{i=0}^{+\infty} \frac{\tau^i}{i!} Z^{[i]} + \tau^{s+1}A(Z) + \tau^{s+2}B(Z) + \tau^{s+3}C(Z) + \tau^{s+4}D(Z) + O(\tau^{s+5}) \tag{4}$$

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2) More generally, one can use an STO to characterize any DS compatible with (1), and obviously the STO can be written in form (4).

(where $Z^{[0]}=Z$, $Z^{[1]}=f(Z)$, $Z^{[k+1]}=\frac{\partial Z^{[k]}}{\partial Z}Z^{[1]}$ for $k=1,2,\cdots$) with complete formulae for calculation of $A(Z)$, $B(Z)$, $C(Z)$ and $D(Z)$.

Thus, the STO G^τ satisfying equation (3) completely characterizes the LMSM (2) as: $Z_1 = G^\tau(Z_0), \cdots, Z_m = G^\tau(Z_{m-1}) = [G^\tau]^m(Z_0), \cdots$.

In the present paper, we study the composition of any two STO's, the inverse operator and the conjugate operator of STO for any DS. In §2, for any two DS's of order $w-1 (w \geq 2)$, we expand the composition of their STO's up to $O(\tau^{w+5})$ (Theorem 1). In §3 and §4, we do the same things for the inverse operator and the conjugate operator of STO for any DS, respectively (Theorems 2-3). And examples for calculation for these three cases (composition, inverse, conjugation) are given in §3 (Examples 1-2) and §4 (Remark 1) respectively.

2. COMPOSITION OF TWO STEP-TRANSITION OPERATORS

Theorem 1. *The composition of two STO's ($w \geq 2$, λ and μ are real numbers)*

$$E^{\mu\tau}(Z) = \sum_{i=0}^{+\infty} \frac{(\mu\tau)^i}{i!} Z^{[i]} + \tau^w B + \tau^{w+1} B_1 + \tau^{w+2} B_2 + \tau^{w+3} B_3 + \tau^{w+4} B_4 + O(\tau^{w+5}) \tag{5}$$

and

$$F^{\lambda\tau}(Z) = \sum_{j=0}^{+\infty} \frac{(\lambda\tau)^j}{j!} Z^{[j]} + \tau^w A + \tau^{w+1} A_1 + \tau^{w+2} A_2 + \tau^{w+3} A_3 + \tau^{w+4} A_4 + O(\tau^{w+5}) \tag{6}$$

can be expressed as follows:

$$\begin{aligned}
& E^{\mu\tau} \circ F^{\lambda\tau}(Z) \\
&= \sum_{l=0}^{+\infty} \frac{(\lambda+\mu)^l \tau^l}{l!} Z^{[l]} + \tau^w \{A+B\} + \tau^{w+1} \left\{A_1 + \mu Z_z^{[1]} A + \lambda B_z Z^{[1]} + B_1\right\} \\
&\quad + \tau^{w+2} \Big\{A_2 + \mu Z_z^{[1]} A_1 + \lambda\mu Z_{z^2}^{[1]} Z^{[1]} A + \frac{\mu^2}{2} Z_z^{[2]} A \\
&\qquad\qquad + \frac{\lambda^2}{2} B_z Z^{[2]} + \frac{\lambda^2}{2} B_{z^2} (Z^{[1]})^2 + \lambda (B_1)_z Z^{[1]} + B_2\Big\} \\
&\quad + \tau^{2w} \{B_z A\} \\
&\quad + \tau^{w+3} \Big\{A_3 + \mu Z_z^{[1]} A_2 + \lambda\mu Z_{z^2}^{[1]} Z^{[1]} A_1 + \frac{\lambda^2\mu}{2} Z_{z^2}^{[1]} Z^{[2]} A + \frac{\lambda^2\mu}{2} Z_{z^3}^{[1]} (Z^{[1]})^2 A \\
&\qquad\qquad + \frac{\mu^2}{2} Z_z^{[2]} A_1 + \frac{\lambda\mu^2}{2} Z_{z^2}^{[2]} Z^{[1]} A + \frac{\mu^3}{6} Z_z^{[3]} A + \frac{\lambda^3}{6} B_z Z^{[3]} + \frac{\lambda^3}{2} B_{z^2} Z^{[1]} Z^{[2]} \\
&\qquad\qquad + \frac{\lambda^3}{6} B_{z^3} (Z^{[1]})^3 + \frac{\lambda^2}{2} (B_1)_z Z^{[2]} + \frac{\lambda^2}{2} (B_1)_{z^2} (Z^{[1]})^2 + \lambda (B_2)_z Z^{[1]} + B_3\Big\} \\
&\quad + \tau^{2w+1} \left\{\frac{\mu}{2} Z_{z^2}^{[1]} A^2 + B_z A_1 + \lambda B_{z^2} Z^{[1]} A + (B_1)_z A\right\} \qquad\qquad (7) \\
&\quad + \tau^{w+4} \Big\{A_4 + \mu Z_z^{[1]} A_3 + \lambda\mu Z_{z^2}^{[1]} Z^{[1]} A_2 + \frac{\lambda^2\mu}{2} Z_{z^2}^{[1]} Z^{[2]} A_1 + \frac{\lambda^3\mu}{6} Z_{z^2}^{[1]} Z^{[3]} A \\
&\qquad\qquad + \frac{\lambda^2\mu}{2} Z_{z^3}^{[1]} (Z^{[1]})^2 A_1 + \frac{\lambda^3\mu}{2} Z_{z^3}^{[1]} Z^{[1]} Z^{[2]} A + \frac{\lambda^3\mu}{6} Z_{z^4}^{[1]} (Z^{[1]})^3 A + \frac{\mu^2}{2} Z_z^{[2]} A_2
\end{aligned}$$