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TWO-SCALE CURVED ELEMENT METHOD FOR ELLIPTIC PROBLEMS WITH SMALL PERIODIC COEFFICIENTS*1)

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Abstract

This paper is concerned with the second order elliptic problems with small periodic coefficients on a bounded domain with a curved boundary. A two-scale curved element method which couples linear elements and isoparametric elements is proposed. The error estimate is obtained over the given smooth domain. Furthermore an additive Schwarz method is provided for the isoparametric element method.

Key words: Two-scale, Curred element, Small periodic coefficients.

1. Introduction

Multiscale phenomenon is often encountered in science and engineering. Typical examples include composite materials and flows in porous media. They are usually described by partial differential equations with highly oscillatory coefficients. Solving these problems by standard element methods is difficult because achieving an approximate solution needs very fine triangulation in general and hence tremendous amount of computer memory and CPU time. Thus it is desirable to have a numerical method that can capture the effect of small scales on large scales without resolving the small scale details.

Two-scale method is a very promising method for solving the above problems (see [4] [5] [6], and references therein). It couples macroscopic scale and microscopic scale together, and not only reflects the global mechanical and physical properties of structure, but also the effect of micro-configuration of composite materials and flows. Using this method, we can slove elliptic problems with small periodic coefficients by solving a homogenization problem with coarse meshes in whole domain and a periodic problem with fine meshes only in one small periodic subdomain.

The objective of this paper is to study the elliptic problems with small periodic coefficients on a bounded domain with a curved boundary. A dual coupled expression is used to approximate the exact solution. Since the homogenization problem is solved with coarse meshes in whole smooth domain, while the periodic problem is solved with fine meshes only in one small periodic subdomain, in order to match the errors of two problems, it is natural to solve the homogenization problem using high order elements and the periodic problem using low order elements. If we use straight side element method to solve the homogenization problem, the smooth domain is approximated by a ploygonal domain. In this case, the error is not optimal, since the best error in the H^1 norm is $O(h_0^{3/2})$, where parameter h_0 is the mesh size (see [11]).

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To overcome this shortcoming, we use isoparametric element method to solve the homogenization problem on smooth domain. But if one uses isoparametric element method in the usual way as in [2], the approximate solutions and the error estimates can be obtained only over an approximate domain Ω_{h_0}. In general, the approximate domain Ω_{h_0} is different from the given smooth domain Ω. To obtain the approximate solutions and the error estimates over the given smooth domain, we use the method given in [9] to define isoparametric element space. Based on this idea, the error of two-scale element method is derived over the given smooth domain. Finally, an additive Schwarz method is proposed for the isoparametric element method. Note that for isoparametric elements both triangulations and finite element spaces are nonnested. Moreover isoparametric element spaces do not contain usual linear conforming element space which is defined on same mesh or coarse meshes in a natural way as a subspace of H_0^1. So we choose a special linear conforming element space as coarse mesh space (for details see section 4).

The remainder of this paper is outlined as follows. Section 2 presents the continuous problems and some notations. Section 3 gives the two-scale curved element method and estimates the error. Section 4 provides an additive Schwarz method.

In this paper, C (with or without subscripts) denotes a generic positive constant with different values in different contexts. For any domain D, we use Sobolev space $W_p^m(D)$ with Sobolev norm $\|\cdot\|_{W_p^m(D)}$ and seminorm $|\cdot|_{W_p^m(D)}$ (see [1]). If $D=\Omega$, we omit D. Moreover if $D=\Omega$ and $p=2$, we denote the usual L^2 inner product by $(\cdot,\cdot)$, the Sobolev norm by $\|\cdot\|_m$ and seminorm by $|\cdot|_m$. Also we use Einstein summation notation, i.e., repeated index indicates to sum.

2. Preliminaries

Consider the following elliptic boundary value problem on a bounded domain $\Omega \subset \mathcal{R}^2$ with a sufficiently smooth curved boundary $\Gamma = \partial\Omega$:

$$\begin{cases} L^\epsilon u^\epsilon \equiv -\bigtriangledown\cdot(a^\epsilon \bigtriangledown u^\epsilon) = f, & \text{in} \quad \Omega, \\ u^\epsilon = 0, & \text{on} \quad \Gamma, \end{cases} \tag{2.1}$$

where $a^\epsilon = (a_{ij}^\epsilon(x))$ is a bounded symmetric positive definite matrix with small period ϵ, and f is a sufficiently smooth function.

Let $y=\frac{x}{\epsilon}$ and $a=(a_{ij}(y))=(a_{ij}^\epsilon(x))$, then $a_{ij}(y)$ is a periodic function with period 1. Let $Q=(0,1)\times(0,1)$. Assume $a_{ij}(y)\in W_\infty^1(Q)$. First we introduce a periodic function $N_k(y)$ which is the solution of the following equation

$$\begin{cases} -\frac{\partial}{\partial y_i}(a_{ij}\frac{\partial N_k}{\partial y_j}) = \frac{\partial}{\partial y_i}a_{ik}, & \text{in} \quad Q, \\ N_k = 0, & \text{on} \quad \partial Q. \end{cases} \tag{2.2}$$

From [8] we know that problem (2.2) has H^2 regularity, i.e., problem (2.2) has a solution $N_k \in H^2(Q)$ satisfying

$$\|N_k\|_{H^2(Q)} \le C\|\frac{\partial}{\partial y_i}a_{ik}\|_{L^2(Q)}.$$

Then we define a constant matrix $a^0=(a_{ij}^0)$ by

$$a_{ij}^0 = \int_Q (a_{ij} + a_{ik}\frac{\partial N_j}{\partial y_k})dy.$$