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EFFICIENT SIXTH ORDER P-STABLE METHODS WITH MINIMAL LOCAL TRUNCATION ERROR FOR $y'' = f(x, y)$ [*1)]

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Abstract

A family of symmetric (hybrid) two step sixth P-stable methods for the accurate numerical integration of second order periodic initial value problems have been considered in this paper. These methods, which require only three (new) function evaluation per iteration and per step integration. These methods have minimal local truncation error (LTE) and smaller phase-lag of sixth order than some sixth orders P-stable methods in [1-3,10-11]. The theoretical and numerical results show that these methods in this paper are more accurate and efficient than some methods proposed in [1-3,10].

Key words: Second order periodic initial value problems, P-stable, Phase-lag, Local truncation error.

1. Introduction

We consider a class of direct hybrid methods proposed in [1] for solving the second order initial value problem

$$y'' = f(t, y), \qquad y(0), y'(0) \quad given \tag{1.1}$$

The basic method has the form

$$\begin{aligned} &y_{n+1} - 2y_n + y_{n-1} \\ =&h^2 \left\{\beta_0 (f_{n+1} + f_{n-1}) + \gamma f_n + \beta_1 (f_{n+\alpha_1} + f_{n-\alpha_1}) + \beta_2 (f_{n+\alpha_2} + f_{n-\alpha_2})\right\} \end{aligned} \tag{1.2a}$$

$$y_{n\pm\alpha_1} = A_\pm y_{n+1} + B_\pm y_n + C_\pm y_{n-1} + h^2 (S_\pm f_{n+1} + Q_\pm f_n + U_\pm f_{n-1}) \tag{1.2b}$$

$$\begin{aligned} y_{n\pm\alpha_2} = &R_\pm y_{n+1} + L_\pm y_n + T_\pm y_{n-1} \\ &+ h^2 (Y_\pm f_{n+1} + V_\pm f_n + W_\pm f_{n-1} + Z_\pm f_{n+\alpha_1} + X_\pm f_{n-\alpha_1}) \end{aligned} \tag{1.2c}$$

and

$$f_n = f(t_n, y_n), \quad f_{n\pm1} = f(t_n \pm h, y_{n\pm1}),$$
$$f_{n\pm\alpha_1} = f(t_n \pm \alpha_1 h, y_{n\pm\alpha_1}), \quad f_{n\pm\alpha_2} = f(t_n \pm \alpha_2 h, y_{n\pm\alpha_2}).$$

Here $t_n = nh$ and we define $t_{n\pm\alpha_i} = t_n \pm \alpha_i h, i = 1, 2$ and n=0,1,2,3,... Several authors (for example, Cash[1,2], Chawla and Rao[3]) have derived sixth order methods of the form(1.2) which are P-stable (see Lambert and Watson[8]). The methods proposed by Cash [1] require five function evaluations per iteration, in general. Cash [2] and Voss and Serbin[12] shown how the number of function evaluation may be reduced to four per iteration. The method proposed in [2] is obtained by choosing $\alpha_2 = 0$ and requiring the points $(t_n \pm \alpha_2 h, y_{n\pm\alpha_2})$ to be coincident. For the method proposed by Chawla and Rao [3], the number of function

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evaluations per iteration is reduced to three by requiring that $y_{n-\alpha_1}$ and $y_{n-\alpha_2}$are independent of y_{n+1}. This implies that$f(t_n - \alpha_1 h, y_{n-\alpha_1})$ and $f(t_n - \alpha_2 h, y_{n-\alpha_2})$ must be computed once per step rather than once per iteration. Finally, Thomas [10], Khiyal and Thomas [7], Khiyal [6] have derived sixth order P-stable, three function evaluation methods of the form (1.2), for which the iteration matrix is a true perfect cube. Here, in section 2, we have given the order condition of sixth order accuracy with form (1.2) and their local truncation error.

In section 3, we introduce two classes of sixth order accuracy, sixth order phase lag, P-stable methods which require only three function evaluation per iteration and per step integration. By choosing free parameters, so that these methods with minimal local truncation error. By computing the LTE of Cash [2], Chawla and Rao [3] and Thomas [10], we know that the efficient sixth order P-stable methods in this paper are smaller of LTE and smaller phase-lag than these methods of Cash [2], Chawla and Rao [3] and Thomas [10].

In section 4, we discuss the implementation of these methods in this paper and numerical illustration on one simple problem.

2. Basic Theory

Thomas [9] have shown that for methods of the form (1.2) applied to the scalar test equation

$$y'' + \lambda^2 y = \omega e^{ivt} \tag{2.1}$$

withλ, ω and v real, the numerical forced oscillation is in phase (Gladwell and Thomas [5]) with its analytical counterpart if and only if

$$\begin{gathered} C_+ + C_- = A_+ + A_-, S_+ + S_- = U_+ + U_-, T_+ + T_- = R_+ + R_-, \\ X_+ + X_- = Z_+ + Z_-, W_+ + W_- = Y_+ + Y_- \end{gathered} \tag{2.2}$$

and they are sixth order accurate if and only if

$$\begin{gathered}
\beta_0 = \frac{1}{12} - \beta_1\alpha_1^2 - \beta_2\alpha_2^2, \qquad \gamma = 1 - 2\beta_0 - 2\beta_1 - 2\beta_2, \\
\beta_1\alpha_1^2 + \beta_2\alpha_2^2 = \frac{1}{20} + \beta_1\alpha_1^4 + \beta_2\alpha_2^4, \\
A_+ + B_+ + C_+ = 1, \qquad A_- + B_- + C_- = 1, \\
A_+ - C_+ = \alpha_1, A_- - C_- = -\alpha_1, \\
A_+ + C_+ + 2(S_+ + Q_+ + U_+) = \alpha_1^2, \\
A_- + C_- + 2(S_- + Q_- + U_-) = \alpha_1^2, \\
R_+ + L_+ + T_+ = 1, R_- + L_- + T_- = 1, \\
R_+ - T_+ = \alpha_2, R_- - T_- = -\alpha_2, \\
R_+ + T_+ + 2(Y_+ + W_+ + V_+ + Z_+ + X_+) = \alpha_2^2, \\
R_- + T_- + 2(Y_- + W_- + V_- + Z_- + X_-) = \alpha_2^2,
\end{gathered}$$

$$\begin{aligned}
&\beta_1\alpha_1 \left[\frac{1}{12}(A_- - C_+) + (S_- - U_+)\right] \\
&+\beta_2\alpha_2 \left[\frac{1}{12}(R_- - T_+) + (Y_- - W_+) + (Z_- - X_+)\alpha_1^2\right] = 0,
\end{aligned}$$