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CHEBYSHEV SPECTRAL-FINITE ELEMENT METHOD FOR TWO-DIMENSIONAL UNSTEADY NAVIER-STOKES EQUATION[*1)]

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Abstract

A mixed Chebyshev spectral-finite element method is proposed for solving two-dimensional unsteady Navier-Stokes equation. The generalized stability and convergence are proved. The numerical results show the advantages of this method.

Key words: Navier-Stokes equation, Chebyshev spectral-finite element method.

1. Introduction

Spectral method has been used successfully in computational fluid dynamics. For semi-periodic problems, we can use mixed Fourier-Chebyshev spectral method, Fourier spectral-finite difference method and Fourier spectral-finite element method (see[1–5]). As we know, many problems are fully non-periodic. But the sections of domains might be rectangular in certain directions. For example, the fluid flow in a cylindrical container. So we proposed Chebyshev spectral-finite element method(see[6]). In this paper, we develop mixed Chebyshev spectral-finite element method for two-dimensional unsteady Navier-Stokes equation.

2. The Scheme

Let $I_x = \{x \,/\, -1 < x < 1\}$, $I_y = \{y \,/\, 0 < y < 1\}$ and $\Omega = I_x \times I_y$ with the boundary $\partial\Omega$. The speed vector and the pressure are denoted by $U = (U_1, U_2)$ and P respectively. $\nu > 0$ is the kinetic viscosity. $U_0(x, y)$ and $f(x, y, t)$ are given functions. Let $T > 0, \partial_t = \frac{\partial}{\partial t}, \partial_x = \frac{\partial}{\partial x}$, and $\partial_y = \frac{\partial}{\partial y}$. The Navier-Stokes equation is as follows

$$\begin{cases} \partial_t U + \partial_x(U_1 U) + \partial_y(U_2 U) + \nabla P - \nu \nabla^2 U = f, & \text{in } \Omega \times (0, T], \\ \nabla^2 P + \Phi(U) = \nabla \cdot f, & \text{in } \Omega \times (0, T], \\ U \mid_{t=0} = U_0, & \text{in } \Omega \bigcup \partial\Omega \end{cases} \tag{2.1}$$

where

$$\Phi(U) = 2(\partial_y U_1 \partial_x U_2 - \partial_x U_1 \partial_y U_2).$$

Suppose that the boundary is a non-slip wall and so $U = 0$ on $\partial\Omega$. There is no boundary condition for the pressure. But if we use the second equation of (2.1) to evaluate the pressure,

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then we need a non-standard boundary condition. We assume approximately that $\frac{\partial P}{\partial n} = 0$ on $\partial\Omega$. For fixing the value of pressure, we require that

$$\mu(P,t) \equiv \int\int_{\Omega} P(x,y,t)\,\mathrm{d}x\mathrm{d}y = 0, \qquad \forall t \in [0,T].$$

Clearly for each time t and U, the second equation of (2.1) is a Neumann problem for P. It can be verified that $\mu(\nabla \cdot f - \Phi(U), t) \equiv 0$ and so this problem is consistent(see[7]). The main advantage of this model is that the derivation of the second formula of (2.1) implies the incompressible condition automatically.

Let $\mathcal{D}$ be an interval (or a domain) in R^1(or R^2). $L^2(\mathcal{D}), H^r(\mathcal{D})$ and $H_0^r(\mathcal{D})(r > 0)$ denote the usual Hilbert spaces with the usual inner products and norms. We also define

$$L_0^2(\mathcal{D}) = \{\eta \in L^2(\mathcal{D}) \,/\, \int_{\mathcal{D}} \eta \,\mathrm{d}\mathcal{D} = 0\}.$$

Let $\omega(x) = (1-x^2)^{-\frac{1}{2}}$ and

$$(u,v)_{\omega,I_x} = \int_{I_x} uv\omega \,\mathrm{d}x, \qquad \|v\|_{\omega,I_x} = (v,v)^{\frac{1}{2}}_{\omega,I_x},$$

$$L^2_\omega(I_x) = \{v \,/\, v \ \text{ is measurable and } \|v\|_{\omega,I_x} < \infty\}.$$

Furthermore

$$(u,v)_\omega = \int\int_{\Omega} uv\omega \,\mathrm{d}x\mathrm{d}y, \quad \|v\|_\omega = (v,v)^{\frac{1}{2}}_\omega,$$

$$L^2_\omega(\Omega) = \{v \,/\, v \ \text{ is measurable and } \|v\|_\omega < \infty\}.$$

Now we construct the scheme. For any positive integer N, we denote by $\mathcal{P}_N$ the set of all polynomials of degree $\leq N$, defined on R^1. Let

$$V_N(I_x) = \{v(x) \in \mathcal{P}_N \ / \quad v(-1) = v(1) = 0\},$$

$$W_N(I_x) = \{v(x) \in \mathcal{P}_N \ / \quad \frac{dv}{dx}(-1) = \frac{dv}{dx}(1) = 0\}.$$

Next, we divide I_y into M_h subintervals with the nodes $0 = y_0 < y_1 < \cdots < y_{M_h} = 1$. Let $I_l = (y_{l-1}, y_l), h_l = y_l - y_{l-1}$, $h = \max\limits_{1\leq l\leq M_h} h_l$ and $h' = \min\limits_{1\leq l\leq M_h} h_l$. Assume that there exists a positive constant d independent of the divisions of I_y, such that $h/h' \leq d$. Let

$$\tilde{S}_h^k(I_y) = \{v(y) \,/\, v(y)\,|_{I_l} \in \mathcal{P}_k, 1 \leq l \leq M_h\}, \quad S_h^k(I_y) = \tilde{S}_h^k(I_y) \bigcap H_0^1(I_y).$$

The trial function space $X^k_{N,h}(\Omega)$ for the speed and the trial function space $Y^k_{N,h}(\Omega)$ for the pressure are defined by

$$X^k_{N,h}(\Omega) = V_N(I_x) \otimes S_h^k(I_y), \quad Y^k_{N,h}(\Omega) = \{W_N(I_x) \otimes (\tilde{S}_h^k(I_y) \bigcap H^1(I_y))\} \bigcap L_0^2(\Omega).$$

In addition, let

$$Z^k_{N,h}(\Omega) = \{\mathcal{P}_{N-2}(I_x) \otimes (\tilde{S}_h^k(I_y) \bigcap H^1(I_y))\} \bigcap L_0^2(\Omega).$$

We denote by P_N^0 the $L^2_\omega(I_x)-$ orthogonal projection from $L^2_\omega(I_x)$ onto $V_N(I_x)$, Π_h^k is the piecewise Lagrange interpolation of order $k \geq 1$, from $C(\bar{I}_y)$ onto $\tilde{S}_h^k(I_y) \bigcap H^1(I_y)$. Furthermore let $P_{N,h} : L^2_\omega(\Omega) \to X^k_{N,h}(\Omega)$ be the orthogonal projection, i.e., for any $v \in L^2_\omega(\Omega)$, the projection $P_{N,h}v \in X^k_{N,h}(\Omega)$ and

$$(v - P_{N,h}v, u)_\omega = 0, \quad \forall u \in X^k_{N,h}(\Omega).$$

Let τ be the mesh size in time t and $S_\tau = \{t = l\tau \ / \ 0 \leq l \leq [\frac{T}{\tau}]\}$. Let

$$u_t(t) = \frac{1}{\tau}(u(t+\tau) - u(t)).$$