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GLOBAL FINITE ELEMENT NONLINEAR GALERKIN METHOD FOR THE PENALIZED NAVIER-STOKES EQUATIONS*1)

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Abstract

A global finite element nonlinear Galerkin method for the penalized Navier-Stokes equations is presented. This method is based on two finite element spaces X_H and X_h, defined respectively on one coarse grid with grid size H and one fine grid with grid size $h << H$. Comparison is also made with the finite element Galerkin method. If we choose $H = O(\varepsilon^{-1/4}h^{1/2})$, $\varepsilon > 0$ being the penalty parameter, then two methods are of the same order of approximation. However, the global finite element nonlinear Galerkin method is much cheaper than the standard finite element Galerkin method. In fact, in the finite element Galerkin method the nonlinearity is treated on the fine grid finite element space X_h and while in the global finite element nonlinear Galerkin method the similar nonlinearity is treated on the coarse grid finite element space X_H and only the linearity needs to be treated on the fine grid increment finite element space W_h. Finally, we provide numerical test which shows above results stated.

Key words: Nonlinear Galerkin method, Finite element, Penalized Navier-Stokes equations.

1. Introduction

In the numerical simulation of the Navier-Stokes equations one encounters three serious difficulties in the case of large Reynolds numbers: the treatment of the incompressibility condition div$u = 0$, the treatment of the nonlinear terms and the large time integration. For the treatment of the incompressibility condition, one use the penalty method in the case of finite elements [1-2] and for the treatment of the nonlinear terms and the large time integration, one use the nonlinear Galerkin method in the framework of finite elements [3]. However, in this work the finite element nonlinear Galerkin method is only used in the time interval $[t_0, \infty)$ and the finite element Galerkin method is used in the finite time interval $[0, t_0]$, $t_0 > 0$ is finite.

Our purpose here is to present a new global finite element nonlinear Galerkin method for the penalized Navier-Stokes equations in the framework of finite elements. This numerical simulation is done in the time interval $[0, \infty)$. Moreover, we analyze the convergence rates of the finite element Galerkin method and the global finite element nonlinear Galerkin method. If $H = O(\varepsilon^{-1/4}h^{1/2})$ is chosen then the global finite element nonlinear Galerkin method provides the same order of approximation as the finite element Galerkin method, where $\varepsilon > 0$ is the penalty parameter. However, in the global finite element nonlinear Galerkin method, the nonlinearity is treated on the coarse grid finite element space X_H and only the linearity is treated on the fine grid increment finite element space W_h; while in the finite element Galerkin method the nonlinearity needs to be treated on the fine grid finite element space X_h. Hence, under the convergence rate of same order, the global finite element nonlinear Galerkin method is much

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cheaper to implement computationally than the finite element Galerkin method. Finally, we provide numerical test which shows the above results stated.

2. The Penalized Navier-Stokes Equations

Let $\Omega \subset R^2$ be a bounded open set with Lipschitz boundary $\Gamma = \partial\Omega$. The Navier-Stokes equations of incompressible flows reads

$$\frac{\partial u}{\partial t} - \nu\Delta u + (u \cdot \nabla)u + \text{grad}p = f, \quad \forall (x,t) \in \Omega \times R^+, \tag{2.1}$$

$$\text{div}u = 0, \quad \forall (x,t) \in \Omega \times R^+, \tag{2.2}$$

where $u = u(x,t)$ is the velocity vector, $p = p(x,t)$ is the pressure, $\nu > 0$ is the kinematic viscosity and f represents the volume driving forces, for simplicity, the constant density ρ was taken equal to 1.

For the penalized equations we suppress the pressure p and the incompressibility equation (2.2) and introduce in (2.1) a penalty term, $\frac{\nu}{\varepsilon}\text{grad div}u$, $\varepsilon > 0$ the penalty parameter. Hence, we obtain the penalized Navier-Stokes equations:

$$\begin{aligned} &\frac{\partial u_\varepsilon}{\partial t} - \nu\Delta u_\varepsilon + (u_\varepsilon \cdot \nabla)u_\varepsilon + \frac{1}{2}(\text{div}u_\varepsilon)u_\varepsilon \\ &\qquad - \frac{\nu}{\varepsilon}\text{grad}\,\text{div}u_\varepsilon = f, \quad \forall (x,t) \in \Omega \times R^+. \end{aligned} \tag{2.3}$$

We have also introduce the supplementary nonlinear term $\frac{1}{2}(\text{div}u_\varepsilon)u_\varepsilon$ which make (2.2) well set.

The equation (2.3) is supplemented by boundary and initial conditions:

$$u_\varepsilon = 0, \qquad \text{on } \Gamma \times R^+, \tag{2.4}$$

$$u_\varepsilon(x,0) = u_0(x), \qquad \forall x \in \Omega. \tag{2.5}$$

We introduce the basic spaces:

$$Y = L^2(\Omega)^2, \quad X = H_0^1(\Omega)^2$$

provided with the scalar products and norms

$$\begin{aligned} (u,v) &= \int_\Omega u(x) \cdot v(x)dx, \quad |u| = (u,u)^{1/2}, && \forall u,v \in Y, \\ ((u,v)) &= \int_\Omega \text{grad}u \cdot \text{grad}v dx, \quad \|u\| = ((u,u))^{1/2}, && \forall u,v \in X. \end{aligned}$$

Moreover, we also introduce the following operators:

$$\begin{aligned} &Au = -\Delta u, \quad B(u,v) = (u \cdot \nabla)v + \frac{1}{2}(\text{div}u)v, \\ &Du = \frac{\nu}{\varepsilon}\text{grad div}u. \end{aligned}$$

It is well-known [1-2] that A is a linear unbounded, self-adjoint positive closed operator with the domain

$$D(A) = X \cap H^2(\Omega)^2,$$

and the inverse A^{-1} of A is a compact self-adjoint operator in Y. Then, we obtain the abstract equation

$$\frac{du_\varepsilon}{dt} + \nu A u_\varepsilon + Du_\varepsilon + B(u_\varepsilon, u_\varepsilon) = f. \tag{2.6}$$