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ALGORITHMS FOR IMPLEMENTATION OF GENERAL LIMIT REPRESENTATIONS OF GENERALIZED INVERSES*

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Abstract

In this paper we investigate three various algorithms for computation of generalized inverses which are contained in the limit expressions $\lim_{z\to 0} V\left(DT+z\mathbf{I}\right)^{-l}U$ and $\lim_{z\to 0} V\left(DT+z\mathbf{I}\right)^{-l}z^{e}$. These algorithms are extensions of the algorithms developed by various authors in [2], [3-4], [7-9], [16-18].

Key words: Generalized inverses, Limit representation, Finite algorithm, Imbedding method.

1. Introduction and Preliminaries

The set of all $m\times n$ complex matrices of rank r is denoted by $\mathbb{C}_r^{m\times n}$. By $\mathbf{I}$ we denote an appropriate identity matrix. Also, $\mathrm{Tr}(A)$ denotes the trace of a square matrix A. By $\mathcal{R}(A)$ and $\mathcal{N}(A)$ are denoted the range and the null space of A, respectively. Finally, $\mathrm{adj}(A)$ and $\det(A)$ denote the adjoint of the matrix A and the determinant of A, respectively.

For any matrix $A\in\mathbb{C}^{m\times n}$ consider the following equations in X:

$$(1)\quad AXA=A,\quad (2)\quad XAX=X,\quad (3)\quad (AX)^*=AX,\quad (4)\quad (XA)^*=XA$$

and if $m=n$, also

$$(5)\quad AX=XA,\quad (1^k)\quad A^{k+1}X=A^k.$$

For a sequence $\mathcal{S}$ of $\{1,2,3,4,5\}$ the set of matrices obeying the equations represented in $\mathcal{S}$ is denoted by $A\{\mathcal{S}\}$. A matrix from $A\{\mathcal{S}\}$ is called an $\mathcal{S}$-inverse of A and denoted by $A^{(\mathcal{S})}$. If X satisfies (1) and (2), it is said to be a reflexive g-inverse of A, whereas $X=A^\dagger$ is said to be the Moore-Penrose inverse of A if it satisfies (1)–(4). The group inverse $A^{\#}$ is the unique $\{1,2,5\}$ inverse of A, and exists if and only if $\mathrm{ind}(A)=\min\{k:\ \mathrm{rank}(A^{k+1})=\mathrm{rank}(A^k)\}=1$. A matrix $G=A^D$ is said to be the Drazin inverse of A if (1^k) (for some positive integer k), (2) and (5) are satisfied.

Let there be given positive definite matrices M and N of the order m and n, respectively. For any $m\times n$ matrix A, the weighted Moore-Penrose inverse of A is the unique solution $X=A^\dagger_{M,N}$ of the matrix equations (1), (2) and the following equations in X:

$$(3M)\qquad (MAX)^*=MAX\qquad\qquad (4N)\qquad (NXA)^*=NXA.$$

In this paper we investigate three methods for implementation of the following limit expressions, related to a given matrix A of the order $m\times n$:

$$L=\lim_{z\to 0} V(DT+z\mathbf{I})^{-l}U,\qquad L_1=\lim_{z\to 0} V(DT+z\mathbf{I})^{-l}z^e,\tag{1.1}$$

where D, T, U and V are appropriate variable complex matrices of the order $q\times p$, $p\times q$, $q\times m$ and $n\times q$, respectively, $l\geq 1$ and e is an arbitrary integer. These limit expressions contain all so far known limit representations of generalized inverses investigated in [1], [5], [6], [8], [10-15], [18-20]. Moreover, in the case $D=U$, $V=\mathbf{I}$ we obtain the limit expression investigated in [16].

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The paper is organized as follows. In the second section we establish and investigate a general imbedding method for computing the generalized inverses included in the limit expressions (1.1). This method deals with a system of first-order ordinary differential equations associated to the matrices $F^l(z) = (\mathrm{adj}(DT + z\mathbf{I}))^l$, $H^l(z) = F^l(z)z^l$ and the scalar $g^l(z) = (\det(DT + z\mathbf{I}))^l$. In certain particular cases we obtain the results originated in [9], [17] and [18].

In the third section is investigated implementation of the limit representations (1.1) by means of several sets of orthogonal vectors. This implementation is an extension of the method introduced in [9] for implementation of the known limit representation of the Moore-Penrose inverse.

In the last section, using a generalization of the method from [8] and [16], we introduce a more condensed form of the Leverrier-Faddeev finite algorithm for computation of various generalized inverses. Introduced algorithm contains known generalizations of the Leverrier-Faddeev algorithm, available in [2], [4], [7-9] and [16-17]. A part of this method which concerns the limit L in the single case $V = \mathbf{I}$, $D = T$ reduces to the known generalization of the Leverrier-Faddeev algorithm, introduced in [16].

2. A Generalized Imbedding Method

In this section we develop a generalization of the imbedding methods, introduced in [9], [17] and [18]. This generalization of the imbedding method can be used in implementation of the limit expressions (1.1). This method is based on the integration of the first-order ordinary differential equations associated to the matrix powers $F^l = F^l(z) = (\mathrm{adj}(DT + z\mathbf{I}))^l$, $H^l = H^l(z) = F^l(z)z^l$ and the scalar $g^l = g^l(z) = (\det(DT + z\mathbf{I}))^l$.

Theorem 2.1. *Consider arbitrary matrices $D \in \mathbb{C}^{q\times p}$, $T \in \mathbb{C}^{p\times q}$, $U \in \mathbb{C}^{q\times m}$ and $V \in \mathbb{C}^{n\times q}$, an integer $l \geq 1$ and an arbitrary integer e. For the matrix $B(z) = DT + z\mathbf{I}$, let the matrices $F(z)$, $H(z)$ and the scalar $g(z)$ are defined by*

$$\begin{aligned} &F = F(z) = \mathrm{adj}(B(z)) = (B_{ij}), \quad H = H(z) = F(z)z, \\ &g = g(z) = \det(B(z)). \end{aligned} \tag{2.1}$$

Then $F^l(z)$, $H^l(z)$ and $g^l(z)$ satisfy the following ordinary differential equations:

$$\begin{aligned} \frac{d(F^l)}{dz} &= lF^l \frac{g^{l-1}\,\mathrm{Tr}(F) - B^{l-1}F^l}{g^l}, \\ \frac{d(g^l)}{dz} &= lg^{l-1}\,\mathrm{Tr}(F), \\ \frac{d(H^l)}{dz} &= z^{l-1}\frac{g^l - lzF^lB^{l-1} - lzg^{l-1}\,\mathrm{Tr}(F)}{g^l}F^l. \end{aligned} \tag{2.2}$$

Assume that the matrices $F^l(z)$, $H^l(z)$ and the scalar $g^l(z)$ satisfy the following initial conditions:

$$F^l(z_0) = (\mathrm{adj}(DT + z_0\mathbf{I}))^l, \quad H^l(z_0) = F^l(z_0)z_0^l, \quad g^l(z_0) = (\det(DT + z_0\mathbf{I}))^l$$

where

$$z_0 > 0, \ |z_0| \leq \min_{z_i\in S}|z_i|, \quad S = \{z_i | z_i > 0 \ \text{is the eigenvalue of} \ -DT\}. \tag{2.3}$$

In this case is

$$\begin{aligned} L &= \lim_{z\to 0} V(DT + z\mathbf{I})^{-l}U \\ &= V \lim_{z\to 0} \frac{F^l(z_0) + l\int_{z_0}^{z} F^l \frac{g^{l-1}\,\mathrm{Tr}(F) - B^{l-1}F^l}{g^l}dz}{g^l(z_0) + l\int_{z_0}^{z} g^{l-1}\,\mathrm{Tr}(F)dz}U. \end{aligned}$$