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THE NUMERICAL SOLUTION OF FIRST KIND INTEGRAL EQUATION FOR THE HELMHOLTZ EQUATION ON SMOOTH OPEN ARCS*[1)]

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Abstract

Consider solving the Dirichlet problem of Helmholtz equation on unbounded region $R^2\backslash\Gamma$ with Γ a smooth open curve in the plane. We use simple-layer potential to construct a solution. This leads to the solution of a logarithmic integral equation of the first kind for the Helmholtz equation. This equation is reformulated using a special change of variable, leading to a new first kind equation with a smooth solution function. This new equation is split into three parts. Then a quadrature method that takes special advantage of the splitting of the integral equation is used to solve the equation numerically. An error analysis in a Sobolev space setting is given. And numerical results show that fast convergence is clearly exhibited.

Key words: Helmholtz equation, Quadrature method.

1. Introduction

The mathematical tratement of the scattering of time-harmonic acoustic or electromagnetic waves by an infinitely long semi-cylindrical obstacle with a smooth open contour cross-section $\Gamma \subset R^2$ leads to unbounded boundary value problems for the Helmholtz equation [3]

$$\left\{ \begin{array}{rcl} \Delta w + k^2 w &=& 0, \text{ in } R^2\backslash\Gamma, \\ w &=& g, \text{ on } \Gamma, \\ \frac{\partial w}{\partial r} - ikw &=& o(\frac{1}{\sqrt{r}}), \qquad r = |x| \to \infty, \end{array} \right. \tag{1.1}$$

with wave number $k > 0$.

In the single-layer approach one seeks the solution in the form

$$w(x) = \int_\Gamma K_0(|x-y|)\varphi(y)ds_y, y \in R^2\backslash\Gamma, \tag{1.2}$$

where ds_y is the element of arc length, and the fundamental solution to the Helmholtz equation is given by

$$K_0(|x-y|) := \frac{1}{2i} H_0^{(1)}(k|x-y|), x \neq y, \tag{1.3}$$

in terms of the Hankel function $H_0^{(1)}$ of order zero and of the first kind. It is known that

$$H_0^{(1)} = J_0 + iN_0, \tag{1.4}$$

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with Bessel function of order zero J_0 and Neumann function of order zero N_0

$$\begin{cases} J_0(z) &= \sum\limits_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2}(\frac{z}{2})^{2n} \\ N_0(z) &= \frac{2}{\pi}(\ln\frac{z}{2}+C)J_0(z)+\frac{2}{\pi}\sum\limits_{n=1}^{\infty}\{\sum\limits_{m=1}^{\infty}\frac{1}{m}\}\frac{(-1)^{n+1}}{(n!)^2}(\frac{z}{2})^{2n} \end{cases} \tag{1.5}$$

where $C = 0.57721\cdots$ is the Euler's constant.

The single-layer potential (1.2) solves the unbounded Dirichlet problem provided the density φ is a solution of the integral equation

$$\int_\Gamma K_0(|x-y|)\varphi(y)ds_y = g(x), x \in \Gamma, \tag{1.6}$$

This integral equation can be shown to be uniquely solvable provided the homogeneous Dirichlet problem for the case of domain bounded by open arc Γ admits only the trivial solution, that is, if the wave number k is not a Dirichlet eigenvalue for the negative Laplacian for the domain bounded by Γ. These eigenvalues are discrete and accumulate only at infinitely [3].

Let Γ have a parametrization

$$r(x) = (\xi(x), \eta(x)), -1 \le x \le 1, \tag{1.7}$$

with

$$|r'(x)| := \{[\xi'(x)]^2 + [\eta'(x)]\}^{\frac{1}{2}} \neq 0, -1 \le x \le 1, \tag{1.8}$$

To simplify the analysis, assume $r(x)$ is C^∞. Following [2, 14], we make the additional change of variable

$$t = \arccos(x), -1 \le x \le 1. \tag{1.9}$$

The equation (1.6) can now be written as

$$-\frac{1}{\pi}\int_0^\pi u(\tau)K(t,\tau)d\tau = f(\tau), 0 \le t \le \pi, \tag{1.10}$$

with

$$\begin{cases} a(t) = r(\cos t), \\ u(t) = \varphi(a(t))|r'(\cos t)|\sin t, \\ f(t) = g(a(t)), \\ K(t,\tau) = -\pi K_0(|a(t)-a(\tau)|). \end{cases} \tag{1.11}$$

Note that $a \in C^\infty$. From the expansions (1.5) we see that the kernel $K(t,\tau)$ can be written in the form

$$K(t,\tau) = (1 + K_1(t,\tau)|\cos t - \cos\tau|^2)\ln(\frac{2}{e}|\cos t - \cos\tau|) + K_2(t,\tau), \tag{1.12}$$

where

$$K_1(t,\tau) = -\frac{J_0(k|a(t)-a(\tau)|)-1}{|\cos t - \cos\tau|^2}, t \neq \tau, \tag{1.13}$$

$$K_2(t,\tau) = K(t,\tau) - (1 + K_1(t,\tau)|\cos t - \cos\tau|^2)\ln(\frac{2}{e}|\cos t - \cos\tau|), t \neq \tau. \tag{1.14}$$

With the assumption on $r(x)$, it can be shown that $K_1(t,\tau), K_2(t,\tau)$ are infinitely differentiable on t and τ and also 2π-periodic and even with respect to each variable. Furthermore, we have the diagonal terms

$$\begin{cases} K_1(t,t) = \frac{k^2}{4}|r'(\cos t)|, \\ K_2(t,t) = -\frac{\pi}{2i} - C - \ln(\frac{ke}{4}|r'(\cos t)|). \end{cases} \tag{1.15}$$