Journal of Computational Mathematics, Vol.19, No.5, 2001, 459–466.

AN EXTREMAL APPROACH TO BIRKHOFF QUADRATURE FORMULAS*[1)]

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Abstract

As we know, a solution of an extremal problem with Hermite interpolation constraints is a system of nodes of corresponding Gaussian Hermite quadrature formula (the so-called Jacobi approach). But this conclusion is violated for a Birkhoff quadrature formula. In this paper an extremal problem with Birkhoff interpolation constraints is discussed, from which a large class of Birkhoff quadrature formulas may be derived.

Key words: An extremal approach, Birkhoff quadrature formulas.

1. Introduction and Main Results

In this paper we shall use the definitions and notations of [3]. Let $E = (e_{ik})_{i=0,\ k=0}^{m+1\ n}$ be an incidence matrix with entries consisting of zeros and ones and satisfying $|E| := \sum_{i,k} e_{ik} = n+1$ (here we allow a zero row). Furthermore, in what follows we assume that

(A) E satisfies the Pólya condition

$$\sum_{i=0}^{m+1} \sum_{k=0}^{r} e_{ik} \geq r+1, \quad r = 0, 1, ..., n; \tag{1.1}$$

(B) all sequences of E in the interior rows, $0 < i < m+1$, are even.

Let S_m denote the set of points $X = (x_0, x_1, ..., x_m, x_{m+1})$ for which

$$0 = x_0 < x_1 < ... < x_m < x_{m+1} = 1 \tag{1.2}$$

and $\overline{S}_m$ its clousure. If some of the coordinates of $X \in \overline{S}_m$ coincide, E is replaced by its corresponding coalescence [3, p. 27]. Then by the Atkinson-Sharma Theorem [3, p. 10] the pair (E, X) is regular for all $X \in \overline{S}_m$ and the quadrature formula of the form

$$\int_0^1 f(x) dg(x) = \sum_{e_{ik}=1} a_{ik} f^{(k)}(x_i) \tag{1.3}$$

is exact for all $f \in \mathbf{P}_n$, the space of all polynomials of degree at most n, where $g(x)$ is a strictly increasing function.

Among all quadrature formulas particularly interesting is the one which is derived from the extremal problem:

$$\int_0^1 |\Omega(E, X; x)| dg(x) = \min_{Y \in \overline{S}_m} \int_0^1 |\Omega(E, Y; x)| dg(x), \tag{1.4}$$

* Received June 1, 1996, Final revised November 20, 2000.
[1)] Project 19671082 supported by National Natural Science Foundation of China.

where $\Omega(x) := \Omega(E, X; x) = x^{n+1} + \dots$ satisfies

$$\Omega^{(k)}(x_i) = 0, \quad e_{ik} = 1 \quad e_{ik} \in E. \tag{1.5}$$

As pointed out in [1], the quantity of the left side in (1.4) is the major term in the estimate of the error of (1.3). Meanwhile, as we know, a sulution of the extremal problem (1.4) with Hermite interpolation constraints must be the system of nodes of corresponding Gaussian Hermite quadrature formula (1.3) (the so-called Jacobi approach). But this conclusion is not valid for a Birkhoff quadrature formula, the reason is that the basic condition (1.2) may be violated. An important question is whether a solution X of (1.4) satisfies (1.2)? Only few papers discuss this question for a proper Birkhoff quadratrue formula. One of them is given by K. Jetter [2]. The main aim of this paper will give a sufficient condition that a solution X of (1.4) satisfy (1.2), from which a class of Birkhoff quadrature formulas may be derived. To state our results, for each i, $0 \le i \le m+1$, let k_i denote the smallest index k such that $e_{ik} = 1$ (when the i-th row is a zero row, we assume $k_i = +\infty$). Put

$$\mu_i = \min\{k_i, k_{i+1}\}, \quad n_i = \sum_{k=0}^{n} (e_{ik} + e_{i+1,k}), \quad i = 0, 1, \dots, m. \tag{1.6}$$

The main result in this paper is the following

Theorem. *Let an incidence matrix E satisfy the conditions* (A) *and* (B). *Assume that*

(C) *there is an index I, $0 \le I \le m$, such that*

$$\begin{cases} \mu_{i+1} \le \mu_i, & i < I, \\ \mu_i \le \mu_{i+1}, & i \ge I; \end{cases} \tag{1.7}$$

(D) *for each i, $1 \le i \le m-1$,*

$$\sum_{k=\mu_i}^{\mu_i + r} (e_{ik} + e_{i+1,k}) \ge r + 1, \quad r = 0, 1, \dots, n_i - 1 \tag{1.8}$$

and

$$e_{i,\mu_i+n_i-1} = e_{i+1,\mu_i+n_i-1} = 0. \tag{1.9}$$

Then each solution of (1.4) satisfies (1.2).

Moreover, (1.3) is exact for all $f \in \mathbf{P_n}$, where

$$\sum_{\substack{k=0 \\ e_{ik}=1}}^{n} a_{ik} \Omega^{(k+1)}(E, X; x_i) = 0, \quad i = 1, \dots, m. \tag{1.10}$$

A specail case of this theorem when each interior row of E contains only one sequence can be found in [2, Theorem 5.1].

In the next section we derive some auxiliary lemmas. The proof of the theorem is put in Section 3. In the last section we give a remark. Our proofs use many ideas of [1,2].

2. Auxiliary Lemmas

First we derive some properties of the polynomials $\Omega(x)$.

Lemma 1. [2, Lemma 2.2] *Let E satisfy the conditions (A) and (B).*

(a) *The polynomials $\Omega(x)$ depend continuously on $X \in \overline{S}_m$.*

(b) *For all $X \in \overline{S}_m$ we have $(-1)^\epsilon \Omega(x) \ge 0$, $x \in [0,1]$, where ϵ is the number of entries $e_{ik} = 1$ in the last row of E.*