Journal of Computational Mathematics, Vol.19, No.5, 2001, 449–458.

ON THE L_∞ CONVERGENCE AND THE EXTRAPOLATION METHOD OF A DIFFERENCE SCHEME FOR NONLOCAL PARABOLIC EQUATION WITH NATURAL BOUNDARY CONDITIONS*[1)]

Zheng-su Wan
(*Institute of Applied Physics and Computational Mathematics, P.O. Box 8009, Beijing 100088, China*)

Zhi-zhong Sun
(*Department of Applied Mathematics, Southeast University, Nanjing 210096, China*)

Abstract

In paper [4] (J. Comput. Appl. Math., 76 (1996), 137-146), a difference scheme for a class of nonlocal parabolic equations with natural boundary conditions was derived by the method of reduction of order and the unique solvability and second order convergence in L_2−norm are proved. In this paper, we prove that the scheme is second order convergent in L_∞ norm and then obtain fourth order accuracy approximation in L_∞ norm by extrapolation method. At last, one numerical example is presented.

Key words: Parabolic, Nonlocal, L_∞ convergence, Extrapolation method.

1. Introduction

Nonlocal parabolic equations have many applications. For example, in considering fluid flow in a saturated porous medium, the equation governing the pore presure $p(r,t)$ in an annular cylindrical rock sample is given in [1] as

$$\frac{\partial p}{\partial t}+\frac{2(\nu_\mu-\nu)}{\eta(1-\nu_\mu)}\frac{dC(t)}{dt}=c(\frac{\partial^2 p}{\partial r^2}+\frac{1}{r}\frac{\partial p}{\partial r}),\quad 0<a<r<b,t>0 \tag{1.1}$$

together with

$$C(t)=(b^2-a^2)^{-1}\left[\frac{a^2}{2}p_0+\eta\int_a^b \rho p(\rho,t)d\rho\right] \tag{1.2}$$

where c is a material constant with dimensions of velocity, the coefficient of consolidation, ν,ν_μ are shear coefficients and η a material constant, r denotes radial distance and t time. If t is replaced by ct and $\frac{2(\nu_\mu-\nu)}{\eta(1-\nu_\mu)}$ by q in (1.1), we get

$$\frac{\partial p}{\partial t}+q\frac{dC(t)}{dt}=\frac{\partial^2 p}{\partial r^2}+\frac{1}{r}\frac{\partial p}{\partial r},\quad 0<a<r<b,t>0. \tag{1.3}$$

Various initial and boundary conditions can be considered. The existence and uniqueness of the solution of (1.3) with (1.2) under some initial and boundary conditions are proved in [2] provided

$$q\eta\neq -2. \tag{1.4}$$

* Received December 14, 1998.
[1)] The work is supported by Jiangsu Province's Natural Science Foundation (BK97004) and National Natural Science Foundation (19801007) of China.

It is also pointed out that, in practice, $q > 0$ and $\eta > 0$, so that (1.4) is normally met [2].

Substituting (1.2) into (1.3), we obtain an alternative form of (1.3)

$$\frac{\partial p}{\partial t} - \epsilon \int_a^b \rho \frac{\partial p}{\partial t}(\rho, t) d\rho = \frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r},$$

where

$$\epsilon = -q\eta(b^2 - a^2)^{-1}.$$

(1.4) is equivalent to $\epsilon \neq 2(b^2 - a^2)^{-1}$, or, $\frac{1}{2}(b^2 - a^2)\epsilon \neq 1$. When $q > 0$ and $\eta > 0$, we have $\epsilon < 0$. In the following, we suppose

$$\frac{1}{2}(b^2 - a^2)\epsilon < 1. \tag{1.5}$$

It is always valid when $\epsilon < 0$.

As usual, we write r by x, p by u. Lin, Tait [3] considered the finite difference solution to the nonlocal parabolic equation given by

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{1}{x}\frac{\partial u}{\partial x} + \epsilon \int_a^b \rho \frac{\partial u}{\partial t}(\rho, t) d\rho, \quad 0 < a < x < b, 0 < t \le T, \tag{1.6}$$

subject to suitable initial and boundary conditions. If the initial-boundary conditions are of the form

$$\begin{array}{lll} u(x,0) = \phi(x), & a \le x \le b, & \\ u(a,t) = f(t), & u(b,t) = g(t), & 0 < t \le T, \end{array} \tag{1.7}$$

a backward Euler scheme and a Crank-Nicolson scheme are presented, with the former giving rise to an error $O(\tau + h^2)$ and the latter to an error $O(\tau^2 + h^2)$. If the natural boundary conditions

$$\begin{array}{lll} u(x,0) = \phi(x), & a \le x \le b, & \\ u(a,t) = f(t), & \frac{\partial u}{\partial x}(b,t) + u(b,t) = g(t), & 0 < t \le T \end{array} \tag{1.8}$$

are imposed, a difference scheme whose convergence order is only one in space and in time is presented. Sun [4] continually studied the finite difference solution to (1.6) with (1.8) and constructed a difference scheme by the method of reduction of order. He proved that the difference scheme is uniquely solvable and unconditionally convergent with the convergence order $O(\tau^2 + h^2)$ in energy norm. In this paper, we will prove that Sun's difference scheme is also second order convergent in $L_\infty-$norm and then obtain a fourth order accuracy approximation in $L_\infty-$norm by once extrapolation [5,6]. At last, we present a numerical example.

For generality, instead of (1.6), consider the inhomogeneous equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{1}{x}\frac{\partial u}{\partial x} + \epsilon \int_a^b \rho \frac{\partial u}{\partial t}(\rho, t) d\rho + \Phi(x,t), \quad 0 < a < x < b, 0 < t \le T \tag{1.9}$$

together with (1.8). Let M and K be two positive integers and $h = \frac{b-a}{M}$, $\tau = \frac{T}{K}$. Denote

$$\Omega_h = \{x_i \mid x_i = a + ih, 0 \le i \le M\}, \quad \Omega_\tau = \{t_k \mid t_k = k\tau, 0 \le k \le K\},$$

$$x_{i-1/2} = (x_i + x_{i-1})/2, \qquad t_{k-1/2} = (t_k + t_{k-1})/2.$$

If $u = \{u_i \mid 0 \le i \le M\}$ and $v = \{v_i \mid 0 \le i \le M\}$ are two mesh functions on Ω_h, take the notations

$$\begin{array}{l} u_{i-1/2} = (u_i + u_{i-1})/2, \qquad \delta_x u_{i-1/2} = (u_i - u_{i-1})/h, \\ \delta_x(x_i \delta_x u_i) = (x_{i+1/2}\delta_x u_{i+1/2} - x_{i-1/2}\delta_x u_{i-1/2})/h \\ (u,v) = h\sum_{i=1}^M x_{i-1/2} u_{i-1/2} v_{i-1/2}, \quad \|u\| = \sqrt{(u,u)}, \quad \|u\|_\infty = \max_{0 \le i \le M}|u_i|, \\ \|\delta_x u\| = \sqrt{h\sum_{i=1}^M x_{i-1/2}(\delta_x u_{i-1/2})^2}, \quad \|\delta_x u\|_* = \sqrt{h\sum_{i=1}^M \frac{1}{x_{i-1/2}}(\delta_x u_{i-1/2})^2}. \end{array}$$