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LONG-TIME BEHAVIOR OF FINITE DIFFERENCE SOLUTIONS OF A NONLINEAR SCHRÖDINGER EQUATION WITH WEAKLY DAMPED*1)

Fa-yong Zhang
(*Department of Mathematics, Heilongjiang University, Harbin 150080, China*)

Shu-juan Lu
(*Department of Mathematics, Beijing University of Aeronautics & Astronautics, Beijing 100083, China*)

Abstract

A weakly demped Schrödinger equation possessing a global attractor are considered. The dynamical properties of a class of finite difference scheme are analysed. The exsitence of global attractor is proved for the discrete system. The stability of the difference scheme and the error estimate of the difference solution are obtained in the autonomous system case. Finally, long-time stability and convergence of the class of finite difference scheme also are analysed in the nonautonomous system case.

Key words: Global attractor, Nonlinear Schrödinger equation, Finite difference method, Stibility and convergence.

1. Introduction

The nonlinear schrödinger equation with weakly damped

$$i\frac{\partial u}{\partial t}+\frac{\partial^2 u}{\partial x^2}+g(|u|^2)u+i\alpha u=f \quad x\in\Omega, t>0 \tag{1.1}$$

where $i=\sqrt{-1},\alpha>0$, together with appropriate boundary and initial conditions, is arised in many physical fields. The existence of an attractor is one of the most important characteristics for a dissipative system. The long-time dynamics is completely determined by the attractor of the system. J.M. Ghidaglia[1] studied the long-time behavior of the nonlinear Schrödinger equation (1.1) and proved the existence of a compact global attractor $\mathcal{A}$ in $H^1(\Omega)$ which has the finite Hausdroff and fractal dimension under the conditions (1.4) and (1.5) in the follows. Guo Boling[6] construct the approximate intertial manifolds for the equation (1.1) and the order of approximation of these manifolds to the global attractor were derived. At the same time, a semidiscrete finite difference method of the equation was discussed by Yin Yan[7]. In this paper, completely discrete scheme is discussed by finite diffenerce method for the equation with initial condition

$$u(x,0)=u_0(x),\ x\in\Omega \tag{1.2}$$

and Dirichlet boundary conditions:

$$u(0,t)=u(L,t)=0, t\in R^+, \tag{1.3}$$

where $\Omega=(0,L), f\in L^2(\Omega)$,$g(s)(0\le s<\infty)$ is a real valued smooth function that satisfies

$$\lim_{s\to+\infty}\frac{G_+(s)}{s^3}=0 \tag{1.4}$$

and

$$\lim_{s\to+\infty} sup\frac{h(s)-\omega G(s)}{s^3}\le 0,\ for\ some\ \omega>0. \tag{1.5}$$

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where $h(s) = sg(s), G(s) = \int_0^s g(\sigma)d\sigma$ and $G_+(s) = max\{G(s), 0\}$.

2. Finite Difference Scheme

First, let us divide the domain $Q_\infty = [0, L] \times [0, \infty)$ into small grids by the parallel lines $x = x_j(j = 0, 1, \cdots, J)$ and $t = t_n(n = 0, 1, \cdots)$, where J is a positive integer, $x_j = jh, Jh = L, t_n = n\triangle t(j = 0, 1, \cdots, J; n = 0, 1, \cdots)$, h and $\triangle t$ are the spatial and temporal mesh lengths respectively. Denote the complex value discrete functions on the grid points $x_0, x_1, \cdots, x_J$ by $\phi, \psi, \cdots$. We employ $\triangle_+, \triangle_-$ and δ_h to denote the forward difference, the backward difference and the forward difference quotient operaters respectively, i.e.

$$\triangle_+\phi_j = \phi_{j+1} - \phi_j, \triangle_-\phi_j = \phi_j - \phi_{j-1}, \delta_h\phi_j = \frac{\triangle_+\phi_j}{h},$$

We introduce the discrete L^2 inner product

$$(\phi, \psi)_h = \sum_{j=0}^{J} \phi_j\overline{\psi}_j h$$

and the discrete H^1 inner product

$$(\phi, \psi)_{1,h} = \sum_{j=0}^{J-1} \delta_h\phi_j\overline{\delta_h\psi}_j h,$$

together with the associated norms

$$\|\phi\|_h = (\phi, \phi)_h^{\frac{1}{2}}, \quad \|\phi\|_{1,h} = (\phi, \phi)_{1,h}^{\frac{1}{2}}.$$

Finally, let

$$\|\phi\|_\infty = \max_{0\le j\le J} |\phi_j|.$$

It is convenient to let L_h^2 and H_h^1 be the normed vector space $\{C^{J+1}, \|\bullet\|_h\}$ and $\{C_0^{J+1}, \|\bullet\|_{1,h}\}$ respectively, here $C_0^{J+1} = \{\phi \in C^{J+1}; \phi_0 = \phi_J = 0\}$. We easily obtain by simple calculation

Lemma 2.1. *For any discrete functions $\{\phi_j\}_0^J$ and $\{\psi_j\}_0^J$, there is the relation*

$$\sum_{j=1}^{J-1} \phi_j\triangle_+\triangle_-\psi_j = -\sum_{j=0}^{J-1}(\triangle_+\phi_j)(\triangle_+\psi_j) - \phi_0\triangle_+\psi_0 + \phi_J\triangle_-\psi_J.$$

Lemma 2.2. *For any discrete function $\{\phi_j\}_0^J, \phi_0 = \phi_J = 0$, the following inequality is valid*

$$\|\phi\|_\infty \le \|\phi\|_{1,h}^{\frac{1}{2}}\|\phi\|_h^{\frac{1}{2}}.$$

Proof. From $\phi_0 = \phi_J = 0$ we can see easily the relations

$$\phi_m^2 = \sum_{j=0}^{m-1}(\phi_{j+1} + \phi_j)\delta_h\phi_j h, \ \phi_m^2 = -\sum_{j=m}^{J-1}(\phi_{j+1} + \phi_j)\delta_h\phi_j h,$$

by cauchy inequality, Lemma 2.2 is proved immediately.

Lemma 2.3. *If functions $f(t), f'(t) \in C(R^+) \cap L^2(R^+)$, the following inequality is valid*

$$\triangle t\sum_{k=0}^{\infty} |f(t_k)|^2 \le \triangle t\int_0^\infty |f'(t)|^2 dt + (1 + \triangle t)\int_0^\infty |f(t)|^2 dt.$$

Proof. By the integrating by parts formula, we derive

$$\triangle t|f(t_{k-1})|^2 = \int_{t_{k-1}}^{t_k} 2f(t)f'(t)(t - t_k)dt + \int_{t_{k-1}}^{t_k} f^2(t)dt,$$

then applying Cauchy inequality, summing them up for k from 1 to ∞. This completes the proof.

In order to prove long-time stability and convergence of difference scheme, we need the following discrete Gronwall lemma