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AN ASYMPTOTICAL $O((k+1)n^3L)$ AFFINE SCALING ALGORITHM FOR THE $P_*(k)$-MATRIX LINEAR COMPLEMENTRITY PROBLEM*

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Abstract

Based on the generalized Dikin-type direction proposed by Jansen et al in 1997, we give out in this paper a generalized Dikin-type affine scaling algorithm for solving the $P_*(\kappa)$-matrix linear complementarity problem (LCP). Form using high-order correctors technique and rank-one updating, the iteration complexity and the total computational turn out asymptotically $O((\kappa+1)\sqrt{n}L)$ and $O((\kappa+1)n^3L)$ respectively.

Key words: linear complementarity problem, $P_*(\kappa)$-matrix, affine scaling algorithm

1. Introduction

An LCP is normally for finding vectors $x, s \in \Re^n$ such that:

$$s = Mx + q, \quad x^T s = 0, \quad (x, s) \geq 0. \tag{1}$$

where $q \in \Re^n$ and $M \in \Re^{n\times n}$. An LCP is called monotonic if M is positive semi-definite. In this paper, M is assumed to be a $P_*(\kappa)$-matrix[6][9] i.e. for a $\kappa \geq 0$, M satisfies:

$$(1+4\kappa)\sum_{u_i(Mu)_i\geq 0} u_i(Mu)_i + \sum_{u_i(Mu)_i\leq 0} u_i(Mu)_i \geq 0$$

for any $u \in \Re^n$. Obviously, positive semi-definite matrix is a $P_*(0)$-matrix. It was proved in [10] that M is a $P_*(\kappa)$-matrix iff M is a sufficient[1].

Based on Dikin's approach, Monteiro and Adler proposed in [8] an affine scaling algorithm of primal-dual type for LP whose iteration complexity is $O(nL^2)$, and Jansen et al gave out lately in [3] a primal-dual algorithm whose iteration complexity is $O(nL)$. Later, Jansen et al[5] made an improvement on the complexity of the algorithm given in [3] such that the iteration complexity obtained is asymptotical $O(\sqrt{n}L)$, and the total computational complexity is asymptotical $O(n^{3.5}L)$. Recently, Jansen et al made an unified generalization in [4] of the primal-dual affine scaling directions and, starting from an arbitrary feasible pair (x^0, s^0), produced a generalized Dikin-type affine scaling algorithm for the monotone LCP, of which the iteration complexity is $O(\frac{n}{\rho^2(1-\rho^2)}\log\frac{(x^0)^\top s^0}{\varepsilon})$.

In this paper, we consider the $P_*(\kappa)$-matrix LCP. Based on the generalized Dikin-type direction given in [5], we give out an r-order generalized Dikin-type affine scaling algorithm by using the high-order correctors technique and the rank-one updating, where r is an integer in $[1, \sqrt{n}]$. The iteration complexity of our algorithm is $O((\kappa+1)n^{(r+1)/(2r)}\log\frac{(x^0)^\top s^0}{\varepsilon})$, and the total computational complexity is $O((\kappa+1)(n^{2.5}+rn^2)n^{(r+1)/(2r)}\log\frac{(x^0)^\top s^0}{\varepsilon})$. If $r = \lfloor\sqrt{n}\rfloor$ in

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particular, then the iteration complexity becomes asymptotically $O((\kappa+1)\sqrt{n}\log\frac{(x^0)^\top s^0}{\varepsilon})$, and the total computational complexity bound becomes asymptotically $O((\kappa+1)n^3\log\frac{(x^0)^\top s^0}{\varepsilon})$.

2. An r-Order Algorithm

In this paper, the following notations are adopted: For $u, v \in \Re^n_+$, let $\min(u)$ and $\max(u)$ denote reapectively $\min_{1\leq i\leq n} u_i$ and $\max_{1\leq i\leq n} u_i$, and let uv and $u^h (h \in \Re)$ represent respentively vectors of $\Re^n$ that $(uv)_i = u_i v_i$ and $(u^h)_i = (u_i)^h$.

Denote the set of strict feasible solution $\{(x,s)\in\Re^n\times\Re^n : s = Mx+q, (x,s)>0\}$ by $\mathcal{F}$, and let

$$\mathcal{N}_\infty(\beta) = \{(x,s)\in\mathcal{F} : \|xs-\mu e\|_\infty \leq \beta\mu\}$$

where $\mu = x^T s/n$ and $\beta\in(0,1)$.

In this paper, we assume $\mathcal{F}\neq\emptyset$; thus, the system (1) is solvable[6].

Our algorithm is as follows:

The algorithm is to be initiated from a given pair (x^0, s^0) that satisfies $(x^0, s^0)\in\mathcal{N}_\infty(\beta)$.

Step 0: Set $k := 0$.

Step 1: Set $(x,s) := (x^k, s^k)$. If $x^T s \leq \varepsilon$ ($\varepsilon > 0$ is a pre-set tolerance error), stop.

Step 2: Let $\gamma\in(0,1)$, and choose $(\tilde{x},\tilde{s})\in\Re^n_+\times\Re^n_+$ such that

$$(\tilde{x}_i)^{-1}|x_i-\tilde{x}_i| \leq \gamma \;\text{ and }\; (\tilde{s}_i)^{-1}|s_i-\tilde{s}_i|\leq\gamma \;\text{ for }\; i = 1,2,\cdots,n. \tag{2}$$

Step 3: Let $w = xs$ and $\ell\geq 1$. compute $(d_x^{(1)}, d_s^{(1)})$ from

$$d_s^{(1)} = Md_x^{(1)}, \quad \tilde{s}d_x^{(1)} + \tilde{x}d_s^{(1)} = -\frac{w^{\ell+1}}{\|w^\ell\|}. \tag{3}$$

Step 4: For $j = 2,\cdots,r$, compute $d_x^{(j)}, d_s^{(j)})$ from

$$d_s^{(j)} = Md_x^{(j)}, \quad \tilde{s}d_x^{(j)} + \tilde{x}d_s^{(j)} = -\sum_{t=1}^{j-1} d_x^{(t)} d_s^{(j-t)}. \tag{4}$$

Step 5: Choose a step length $\bar{\alpha} > 0$ such that the new $(x(\bar{\alpha}), s(\bar{\alpha}))$,

$$x(\bar{\alpha}) = x + (1+\gamma)\sum_{j=1}^r \bar{\alpha}^j d_x^{(j)}, \quad s(\bar{\alpha}) = s + (1+\gamma)\sum_{j=1}^r \bar{\alpha}^j d_s^{(j)},$$

is in $\mathcal{N}_\infty^-(\beta)$.

Step 6: Set $(x^{k+1}, s^{k+1}) := (x(\bar{\alpha}), s(\bar{\alpha}))$, $k := k+1$ and go to Step 1.

The quantity $-\frac{w^{\ell+1}}{\|w^\ell\|}$ given in the step 3 (which was first introduced by Jansen et al[4]) is a generalized Dikin-type affine scaling; when $\ell = 0$, this quantity turns out a classical primal-dual affine scaling[10]; when $\ell = 1$, it becomes a primal-dual Dikin affine scaling[3][5].

For the sake of notational simplicity, we omit in the following discussion the superscript k unless otherwise specified.

Let $w = xs$ and $\tilde{w} = \tilde{x}\tilde{s}$. It is not diffcult to obtain the following results by (2).

$$(1+\gamma)^{-2} w_i \leq \tilde{w}_i \leq (1-\gamma)^{-2} w_i; \tag{5}$$

$$(1-\gamma)\tilde{x}\leq x\leq(1+\gamma)\tilde{x}, \quad (1-\gamma)\tilde{s}\leq s\leq(1+\gamma)\tilde{s}; \tag{6}$$

$$0 < 1-\gamma \leq x_i(\tilde{x}_i)^{-1} \leq 1+\gamma, \quad 0 < 1-\gamma\leq s_i(\tilde{s}_i)^{-1}\leq 1+\gamma. \tag{7}$$

Let $x(\alpha) = x + (1+\gamma)\sum_{j=1}^r \alpha^j d_x^{(j)}, s(\alpha) = s + (1+\gamma)\sum_{j=1}^r \alpha^j d_s^{(j)}$, where α is a certain