Journal of Computational Mathematics, Vol.19, No.2, 2001, 157–166.

MATRIX ANALYSIS ON LEADING TERM OF CONDITION NUMBER FOR ADDITIVE SCHWARZ METHODS*[1)]

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Abstract

It is well known the order of preconditioned matrix by using additive Schwarz methods. In order to estimate the resulted PCG iteration counts, the related leading term brfore the order is given in this paper.

Key words: Additive Schwarz method, PCG, Matrix analysis.

1. Introduction

Let us consider the following second order elliptic boundary value problem:

$$\mathcal{L}u = f \quad in \quad \Omega \tag{1}$$

$$u = 0 \quad on \quad \partial\Omega \tag{2}$$

where $\mathcal{L}$ is a self-adjoint positive operator and

$$\Omega \subset \mathcal{R}^d \quad (1 \leq d \leq 3)$$

is a polyhedral domain.

Using weak solution it leads to a discrete equation

$$Au = f \tag{3}$$

with

$$A = (\alpha_{ij}), \quad \alpha_{ij} = \mathcal{A}(\phi_i, \phi_j) \tag{4}$$

where $\{\phi_i\}$ could be nodal basis consisting of piece-wise linear functions or other spline functions. It is well known that the coefficient matrix A is symmetry positive definite matrix with condition number

$$\kappa(A) := \frac{\lambda_{max}(A)}{\lambda_{min}(A)} = O(h^{-2}) \tag{5}$$

Furthermore, under rectangle uniform mesh it is easy to obtain the leading term of Discrete Laplacian condition number before the above order as follows which is independent on dimension.

* Received March 6, 1998; Final revised January 29, 1999.
1) This work was partly supported by National Natural Science Foundation of China and Laboratory LSEC.

$Dimension$	$Scheme$	λ_{max}	λ_{min}	κ
$1-D$	$3-point$	$2-(\pi h)^2$	$(\pi h)^2$	$\frac{2}{\pi^2}h^{-2}$
$2-D$	$5-point$	$4-2(\pi h)^2$	$2(\pi h)^2$	$\frac{2}{\pi^2}h^{-2}$
$3-D$	$7-point$	$6-3(\pi h)^2$	$3(\pi h)^2$	$\frac{2}{\pi^2}h^{-2}$

When we use conjugate gradient algorithm for solving the system, for a given tolerance, the iteration number will proportional to h^{-1}. This convergent rate is really slow for a large scale problems. It is our aim to study how better is better the preconditioner by using additive Schwarz method.

Suppose there are two subdomain partitions, one is without overlapping

$$\Omega = \cup\Omega_i \quad with \quad \Omega_i \cap \Omega_j = \phi \quad if \quad i \neq j$$

and another is with overlapping

$$\Omega = \cup\hat{\Omega}_i \quad with \quad \hat{\Omega}_i \cap \hat{\Omega}_j \neq \phi \quad if \quad i \neq j$$

and denote A_i be a matrix representation which is the restriction of the original operator A over the subdomain Ω_i. Without considering unknowns permutation appropriately, then the related preconditioner can be written as

$$B_i = \begin{bmatrix} A_i^{-1} & 0 \\ 0 & 0 \end{bmatrix}$$

and the corresponding matrix A has the following block form

$$A = \begin{bmatrix} A_i & A_{ij} \\ A_{ij}^t & A_j \end{bmatrix}, \qquad B_i A = \begin{bmatrix} I & A_i^{-1}A_{ij} \\ 0 & 0 \end{bmatrix}$$

Hence, in general we obtain a whole subdomain preconditioner B_s as follows:

$$B_s = \sum B_i, \quad with \quad B_i = R_i^t A_i^{-1} R_i \tag{6}$$

where R_i is a truncated permutation matrix from Ω to the subdomain Ω_i.

Some questions now are arisen on the above algorithm as a PCG preconditioner . What is efficiency of the above parallel ASM ? How fast the related PCG iteration does converge? How does the eigenvalues of preconditioned matrix $B_s A$ distribute ? How to estimate the largest and smallest eigenlavues of $B_s A$?

This estimation can be reduced to special generalized eigen-decomposition

$$Au = \lambda B_s^{-1} u$$

From finite element theory, it has been known by using energy norm estimation

$$\lambda_{min}(B_s A) = O((hH)^{-1}), \quad \lambda_{max}(B_s A) = O(1).$$

where h is the smallest mesh size on subdomains, and H is the subdomain width.

In practical, it is need to estimation the leading term before the order to estimate the PCG iteration counts more precisely for a given tolerance.