Journal of Computational Mathematics, Vol.19, No.1, 2001, 35–40.

A SCALED CENTRAL PATH FOR LINEAR PROGRAMMING*[1)]

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Dedicated to the 80th birthday of Professor Feng Kang

Abstract

Interior point methods are very efficient methods for solving large scale linear programming problems. The central path plays a very important role in interior point methods. In this paper we propose a new central path, which scales the variables. Thus it has the advantage of forcing the path to have roughly the same distance from each active constraint boundary near the solution.

Key words: Central path, Interior point methods, Linear programming.

1. Introduction

Interior point methods are one of the most intensively studied topics in optimization. Thousands of publications have been appeared on interior point methods. A very good recent review is given by [4]. Interior point methods have very good theoretical properties including the nice polynomial complexity property. And more important is that numerous applications have shown that interior point methods are very efficient for solving large sparse linear programming problems. Interior point methods have been proved to be indispensable to semi-definite programming, another class of important optimization problems. Interior point methods have also been applied to nonlinear programming and nonlinear complementary problems. For examples of detailed discussions, please see ([1, 2, 3]).

Path following algorithms are a class of very important interior point methods for linear programming. Consider the following standard linear programming problem

$$\min c^T x \tag{1.1}$$

subject to

$$Ax = b, \qquad x \geq 0, \tag{1.2}$$

where $c \in \Re^n$, $b \in \Re^m$ and $A \in \Re^{m\times n}$. The dual problem for the above linear program can be written as

$$\max b^T y \tag{1.3}$$

subject to

$$A^T y + s = c, \qquad s \geq 0, \tag{1.4}$$

where $y \in \Re^m$ are the dual variables and $s \in \Re^n$ are the slack variables. If both the prime problem (1.1)-(1.2) and the dual problem (1.3)-(1.4) have feasible solutions, then both problems

* Received November 15, 2000.
1) This work is partially supported by Chinese NNSF grants 19731010 and the Knowledge Innovation Program of Chinese Academy of Sciences.

have optimal solutions. And, in this case, for any solution x^* of the primal problem and any solution (y^*, s^*) of the dual problem, we have that

$$c^T x^* = b^T y^*, \tag{1.5}$$

(see, [5]). For any point (x, y, x) that satisfies (1.2) and (1.4), it follows that

$$c^T x - b^T y = (A^T y + s)^T x - (Ax)^T y = s^T x \geq 0. \tag{1.6}$$

Thus, a solution is obtained as long as the complementarity gap $s^T x$ is zero. Because both x and s are nonnegative, the condition $s^T x = 0$ is equivalent to $x_i s_i = 0$ for all $i = 1, ..., n$. Let $X = diag[x_1, x_2, ..., x_n]$, relation $s^T x = 0$ can be expressed as $Xs = 0$. Thus, we can write the optimal conditions in the following form

$$\begin{aligned} Ax &= b && (1.7)\\ A^T y + s &= c && (1.8)\\ Xs &= 0 && (1.9)\\ (x, s) &\geq 0. && (1.10) \end{aligned}$$

Define the set

$$\mathcal{F} = \{(x, y, s)\ Ax = b, A^T y + s = c, x \geq 0, s \geq 0\}, \tag{1.11}$$

which is the direct product of the primal feasible set and the dual feasible set. Interior point methods generate iterate point in the interior of the region $\mathcal{F}$, that is

$$int(\mathcal{F}) = \{(x, y, s)\ Ax = b, A^T y + s = c, x > 0, s > 0\}. \tag{1.12}$$

The central path is defined by the following system

$$\begin{aligned} Ax &= b && (1.13)\\ A^T y + s &= c && (1.14)\\ Xs &= \mu e && (1.15)\\ (x, s) &> 0, && (1.16) \end{aligned}$$

where e is a vector whose elements are all 1, and $\mu > 0$ is a parameter. It is easy to see that system (1.13)-(1.16) is a perturbation of the optimal condition (1.7)-(1.10). Let $(x(\mu), y(\mu), s(\mu))$ be on the central path, it can be shown that $x(\mu)$ is a solution of the penalized problem

$$\min\ c^T x\ -\ \mu \sum_{i=1}^{n} \log(x_i) \tag{1.17}$$

subject to

$$Ax = b. \tag{1.18}$$

Many interior point methods use the central path. Some algorithms explicitly use the central path as they force the interate points to follow the central path. Even for many algorithms that do not use the central path directly in the algorithm statements, the central path is used for convergence analyses (see, [5]).

Because of the importance of the central path in the designs and analyses of interior point methods, we study the center path. From the views of complementary conditions, we propose a new central path. We believe that this path can also be used to construct new interior point methods.

A new central path is derived in the next section, and in Section 3 we proposed two ways to compute search directions based on the new central path and in Section 4 we give a brief discussion on how our ideas can be further extended.