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A MAGNUS EXPANSION FOR THE EQUATION $Y' = AY - YB$ *

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Dedicated to the 80th birthday of Professor Feng Kang

Abstract

The subject matter of this paper is the representation of the solution of the linear differential equation $Y' = AY - YB$, $Y(0) = Y_0$, in the form $Y(t) = \mathrm{e}^{\Omega(t)} Y_0$ and the representation of the function Ω as a generalisation of the classical Magnus expansion. An immediate application is a new recursive algorithm for the derivation of the Baker–Campbell–Hausdorff formula and its symmetric generalisation.

Key words: Geometric integration, Magnus expansions, Baker-Campbell-Hausdorff formula.

1. Introduction

This paper is concerned with the solution of the linear ordinary differential system

$$Y' = AY - YB, \quad t \geq 0, \qquad Y(0) = Y_0, \tag{1.1}$$

where both A and B are Lipschitz functions that map $[0, \infty)$ into M_m, the set of $m \times m$ matrices, and $Y_0 \in \mathrm{M}_m$. The equation (1.1) features in numerous applications and the approximation of its solution is of interest. Moreover, solutions of this equation often display interesting geometry. For example, $B = A$ results in the *isospectral flow*

$$Y' = AY - YA, \quad t \geq 0, \qquad Y(0) = Y_0, \tag{1.2}$$

whose invariants are the eigenvalues of Y_0 and which features in numerous areas of applied mathematics (Zanna 1998). Note that if $Y_0 \in \mathrm{Sym}(m)$, the set of symmetric matrices in M_m, while $A(t) \in \mathfrak{so}(m)$, the Lie algebra of $m \times m$ skew-symmetric matrices, then $Y(t) \in \mathrm{Sym}(m)$ for all $t \geq 0$. Another example is

$$Y' = AY + YA^{\mathrm{T}}, \quad t \geq 0, \qquad Y(0) = Y_0 \in \mathrm{M}_m, \tag{1.3}$$

where $A(t) \in \mathrm{M}_m$, $t \geq 0$. In that case $Y(t)$ evolves on a *congruent orbit*, $Y(t) = Q(t) Y_0 Q^{\mathrm{T}}(t)$, $t \geq 0$.

In principle, we can employ one of two obvious means to convert (1.1) to a 'classical' linear form. Firstly, let $\boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_m$ be the columns of Y and set $\boldsymbol{y} = [\boldsymbol{y}_1^{\mathrm{T}}, \boldsymbol{y}_2^{\mathrm{T}}, \ldots, \boldsymbol{y}_m^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{m^2}$.

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It is easy to verify that $\boldsymbol{y}$ obeys a linear equation of the form $\boldsymbol{y}' = \mathcal{A}\boldsymbol{y}$, where, however, $\mathcal{A} \in \mathrm{M}_{m^2}$. This equation can be solved easily by standard *explicit* methods for ordinary differential equations, except that in that case all the nice qualitative and geometric properties of the original system are likely to be lost (Iserles 2000a, Iserles & Zanna 2000). Implicit classical methods are considerably more expensive, since we need to invert $m^2 \times m^2$ matrices. Moreover, all classical methods are likely to display inferior precision in comparison with Lie-group methods (Iserles 2000b). As soon, however, as we use Lie-group methods, which are also considerably better in respecting underlying structure (Iserles, Munthe-Kaas, Nørsett & Zanna 2000), we need to operate (specifically, evaluate commutators and exponentials) of $m^2 \times m^2$ matrices.

An alternative to solving $\boldsymbol{y}' = \mathcal{A}\boldsymbol{y}$ is to represent the solution of (1.1) in the form

$$Y(t) = X(t)Y_0Z^{-1}(t), \qquad t \geq 0, \tag{1.4}$$

where

$$X' = AX, \quad Z' = BZ, \quad t \geq 0, \qquad X(0) = Z(0) = I.$$

The two linear systems, both involving $m \times m$ matrices, can be solved e.g. by Magnus expansions, thereby retaining important geometric features (Iserles et al. 2000). This results in the representation $Y(t) = \mathrm{e}^{\Omega_1(t)}Y_0\mathrm{e}^{-\Omega_2(t)}$, where

$$\Omega_1' = \mathrm{dexp}_{\Omega_1}^{-1}A, \quad \Omega_2' = \mathrm{dexp}_{\Omega_2}^{-1}B, \quad t \geq 0, \qquad \Omega_1(0) = \Omega_2(0) = O$$

(the 'dexpinv' equation and the Magnus expansion will be introduced formally in Section 2). Hence, the approximation of (1.4) calls for the computation of two Magnus expansions and the evaluation of two matrix exponentials.

In this paper we investigate another approach toward the solution of (1.1). Representing $Y(t) = \mathrm{e}^{\Omega(t)}Y_0$, we seek a *Magnus expansion* of the function Ω. This approach is motivated by three considerations:

1. Provided that integrals can be evaluated exactly (e.g., when A and B have polynomial entries), this approach leads to a method that requires less operations than the approximation of (1.4) to the same order of accuracy.

2. An interesting outcome of this approach and of its comparison with (1.4) is a practical algorithm for the evaluation of the *BCH formula*

$$\mathrm{e}^{tR}\mathrm{e}^{tS} = \mathrm{e}^{\mathrm{bch}(t;R,S)}, \tag{1.5}$$

   where $R, S \in \mathrm{M}_m$ and $|t|$ is sufficiently small, and of its symmetric generalisation.

3. The work of the present paper adds to the evolving theory of Magnus and other Lie-group expansions and highlights their connection with graph theory.

2. The Magnus Expansion of Y

We assume forthwith that Y_0 is nonsingular. Letting $Y(t) = \mathrm{e}^{\Omega(t)}Y_0$ in (1.1) and recalling