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# INFINITE ELEMENT METHOD FOR THE EXTERIOR PROBLEMS OF THE HELMHOLTZ EQUATIONS*[1)]

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**Abstract**

There are two cases of the exterior problems of the Helmholtz equation. If $\lambda \geq 0$ the bilinear form is coercive, and if $\lambda < 0$ it is the scattering problem. We give a new approach of the infinite element method, which enables us to solve these exterior problems as well as corner problems. A numerical example of the scattering problem is given.

*Key words*: Helmholtz equation, Exterior problem, Infinite element method.

## 1. Introduction

The infinite element method has been successfully applied to some boundary value problems of partial differential equations, where the solutions possess corner singular points or the domains are exterior ones. If the equations are invariant under similarity transformation the approaches have been given in [11][13] for singular solutions, and in [12][15][18][19] for the exterior problems. If the equations do not admit the above invariant property, one approach has been given in [14] to deal with the singular solutions to the Helmholtz equation, and another approach has been given in [16] to deal with the singular solutions to more general problems where the coefficients of the equations are allowed to be variable and discontinuous. For details see [17] and the references therein.

For the exterior problems of the equations which are not invariant under similarity transformation the above approaches are not valid. Because in [14] the solutions are expanded into Taylor series about the parameter $\lambda$, and in [16] the exact solution can be divided into two parts, one of which is a solution to an associated equation which is invariant under similarity transformation, and the other one of which is a regular function. Now the solutions are neither analytic nor the sum of these two parts.

We will give a new approach of the infinite element method in this paper, which enables us to deal with these problems, and it is also an efficient approach to solve the singular solutions. Firstly we study the Helmholtz equation, and the infinite element method for the exterior problems of some other equations will be given in separate papers.

The terminology of "infinite element" has been employed by many authors for different methods. For example it is employed in infinitely large elements are used on the neighborhood of the infinity and some special interpolation functions are applied

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in the element to simulate the behavior of the solutions near the infinity. Our approach is different. In our approach the size of elements are finite and the number of elements is infinite. Without any truncation we solve the infinite-by-infinite algebraic systems associated with those elements. In fact for each equation we only solve one equivalent algebraic problem which is of extremely small scale. The formulation and the elements of our method are the same as the finite element method, therefore the rate of convergence is the same even if there are singularities or the domain is infinitely large. In our method no analytic expression of the true solution is needed, so there is no special requirement to the domain or to the equations. For example, this method can be applied to the equations with variable coefficients.

This paper is organized as follows: For completeness we recall the infinite element method for the Laplace equation in §2. To meet the needs of our new approach we study the critical case , where the constant of proportionality tends to one in §3. We derive the equations for the combined stiffness matrix of the infinite element method for the exterior problems of the Helmholtz equation in §4. We give some detail computation for the critical stiffness matrices and mass matrices in §5. Some further discussion on the infinite element algorithm to the Helmholtz equation is carried out in §6. We prove the convergence of the approximate solutions for the case of $\lambda > 0$ in §7. Finally we show one numerical example of the scattering problem in §8.

## 2. The Laplace Equation

Let $\Omega \subset \mathsf{R}^2$ be an exterior domain, the boundary of which is a closed curve $\Gamma_0$. For the sake of simplicity we assume that the origin $o \notin \Omega$ and $\Gamma_0$ is star-shape with respect to the point $o$, that is, all line segments connecting the points of $\Gamma_0$ with $o$ lie outside $\Omega$ entirely. For those domains with complicated shape we can decompose them as $\Omega = \overline{\Omega_0 \bigcup \Omega'}$, where $\Omega_0$ is a bounded domain, $\Omega'$ is an unbounded domain. Usual finite element partition is made on $\Omega_0$ and infinite element partition is made on $\Omega'$. Then we solve an equation which is obtained by assembly of these two. The domain decomposition technique can be applied to this decomposition if one wants. For simplicity we will assume $\Omega = \Omega'$ in the sequel.

We assume that $\Gamma_0$ is a polygonal curve. One parameter $\xi > 1$ is taken. We construct similar figures of $\Gamma_0$ with the center $o$ and the constant of proportionality $\xi, \xi^2, \cdots, \xi^k, \cdots$, denoted by $\Gamma_k$. Let $\xi^k\Omega = \{(x,y); (x,y) \text{ is on the exterior of } \Gamma_k\}$, and $\Omega_k = \xi^{k-1}\Omega \setminus \overline{\xi^k\Omega}$. We make conventional finite element partition on each $\Omega_k$. It is required that the meshes of all subdomains $\Omega_k$ are geometrically similar to each other and the partitions on $\Omega_k$ and $\Omega_{k-1}$ are compatible on $\Gamma_k$. For example we can construct some rays starting from the point $o$ which divide each $\Omega_k$ into some quadrilaterals, then each quadrilateral is further divided into two triangular elements.

We define space $H^{1,*}(\Omega) = \{u \in L^2_{\text{loc}}(\Omega); \|u\|_{1,*} < \infty\}$, where the norm is defined as

$$\|u\|_{1,*} = \left( \int_\Omega \left( |\nabla u(x,y)|^2 + \frac{u^2(x,y)}{r^2 \log^2 r} \right) dxdy \right)^{1/2},$$

where $r = \sqrt{x^2+y^2}$. Then we define infinite element space

$$S(\Omega) = \{u \in H^{1,*}(\Omega); u|_{e_i} \in P_1(e_i), i = 1, 2, \cdots\},$$

where $e_i, i = 1, 2, \cdots$ are elements, and $P_1$ is the set of all polynomials with degree $\leq 1$.