Journal of Computational Mathematics, Vol.18, No.6, 2000, 597–608.

THE SOLVABILITY CONDITIONS FOR THE INVERSE PROBLEM OF BISYMMETRIC NONNEGATIVE DEFINITE MATRICES*1)

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Abstract

$A = (a_{ij}) \in R^{n\times n}$ is termed bisymmetric matrix if

$$a_{ij} = a_{ji} = a_{n-j+1,n-i+1}, \quad i, j = 1, 2 \cdots n.$$

We denote the set of all $n \times n$ bisymmetric matrices by $BSR^{n\times n}$.

This paper is mainly concerned with solving the following two problems:

Problem I. Given $X, B \in R^{n\times m}$, find $A \in P_n$ such that $AX = B$, where $P_n = \{A \in BSR^{n\times n} | \quad x^T Ax \geq 0, \quad \forall x \in R^n\}$.

Problem II. Given $A^* \in R^{n\times n}$, find $\hat{A} \in S_E$ such that

$$\|A^* - \hat{A}\|_F = \min_{A\in S_E} \|A^* - A\|_F,$$

where $\|\cdot\|_F$ is Frobenius norm, and S_E denotes the solution set of problem I.

The necessary and sufficient conditions for the solvability of problem I have been studied. The general form of S_E has been given. For problem II the expression of the solution has been provided.

Key words: Frobenius norm, Bisymmetric matrix, The optimal solution.

1. Introduction

Inverse eigenvalue problem has widely been used in engineering. For example inverse eigenvalue method is a useful means in vibration design and vibration control of flyer. In resent years a serial of good conclussions have been made for inverse eigenvalue problem [4]. Bisymmetric matrices have practical application in civil engineering and vibration engineering. However, inverse problems of bisymmetric matrix have not be concerned yet. In this paper we will discuss this problem.

* Received November 22, 1997.
1)Supported by the National Nature Science Fundation of China.

We denote the real $n \times m$ matrices space by $R^{n\times m}$, and $R^n = R^{n\times 1}$, the set of all matrices in $R^{n\times m}$ with rank r by $R_r^{n\times m}$, the set of all $n \times n$ orthogonal matrices by $OR^{n\times n}$, the set of all $n \times n$ symmetric matrices by $SR^{n\times n}$, the column space, the null space and the Moore–Penrose generalized inverse of a matrix A by $R(A), N(A), A^+$ respectively, the identity matrix of order n by I_n, the Frobenius norm of A by $\|A\|_F$. We define inner product in space $R^{n\times m}$, $(A, B) = \operatorname{tr}(B^T A) = \sum\limits_{i=1}^{n}\sum\limits_{j=1}^{m} a_{ij}b_{ij}, \quad \forall A, B \in R^{n\times m}$. Then $R^{n\times m}$ is a Hilbert inner product space. The norm of a matrix produced by the inner product is Frobenius norm.

Definition 1. $A = (a_{ij}) \in R^{n\times n}$, *if*

$$a_{ij} = a_{ji} = a_{n-j+1,n-i+1}, \qquad i, j = 1, 2, \cdots, n.$$

Then we term A as a bisymmetric matrix. The set of all bisymmetric matrices denoted by $BSR^{n\times n}$.

Let

$$k = [\frac{n}{2}], \qquad \text{[x] is the maximum integer number that is not greater than x.} \tag{1.1}$$

When $n = 2k$,

$$D = \frac{1}{\sqrt{2}}\begin{pmatrix} I_k & I_k \\ S_k & -S_k \end{pmatrix}; \tag{1.2}$$

and when $n = 2k + 1$,

$$D = \frac{1}{\sqrt{2}}\begin{pmatrix} I_k & 0 & I_k \\ 0 & \sqrt{2} & 0 \\ S_k & 0 & -S_k \end{pmatrix}, S_k = \begin{pmatrix} 0 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 0 \end{pmatrix}_{k\times k}. \tag{1.3}$$

It is easy verified that above D are orthogonal.

Definition 2. *$A \in BSR^{n\times n}$ is termed bisymmetric nonnegative definite [positive definite] if $x^T Ax \geq 0 (> 0)$ for every nonzero x in R^n.*

Let

$$P_n = \{A \in BSR^{n\times n} | \quad x^T Ax \geq 0, \quad \forall x \in R^n\}.$$

Now we consider the following problems:

Problem I. Given $X, B \in R^{n\times m}$, find $A \in P_n$ such that

$$AX = B.$$

Problem II. Given $A^* \in R^{n\times n}$, find $\hat{A} \in S_E$ such that

$$\|A^* - \hat{A}\|_F = \min_{A\in S_E} \|A^* - A\|_F,$$

where S_E is the solution set of problem I.

At first, in this paper, we will discuss the geometric constrution of $BSR^{n\times n}$. Then we will give the necessary and sufficient conditions for the solvability of problem I and the expression of the general solution of problem I, and prove that S_E is a closed convex set. At last, we will prove that there exists an unique solution of problem II and give expression of the solution for problem II.