

THE CONVERGENCE OF DURAND-KERNER METHOD FOR SIMULTANEOUSLY FINDING ALL ZEROS OF THE POLYNOMIAL^{*1)}

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Abstract

In this paper we propose the two kinds of different criterions and use them to judge the convergence of Durand-Kerner method and to compare the obtained results with other methods.

Key words: Durand-Kerner method, Convergence, Two kinds of Criterions.

1. Introduction

Consider a monic polynomial of degree n

$$f(z) = z^n + a_1 z^{n-1} + \cdots + a_n := \prod_{j=1}^n (z_j - \zeta_j).$$

To solve all zeros of this polynomial, the common-used is Durand-Kerner method which has been considered simple and effective. For this reason, there are many works (Docev (1962), kerner (1966), Yamamoto (1990) and Zheng (1982)) to discuss the properties of this method. In this paper, we propose the two kinds of criterion and use them to judge the convergence of Durand-Kerner method or to compare with other methods.

Durand-Kerner method is defined by

$$z_i^{(k+1)} = z_i^{(k)} - \frac{f(z_i^{(k)})}{\prod_{j \neq i} (z_i^{(k)} - z_j^{(k)})}, \quad i = 1, 2, \dots, n; \quad k = 0, 1, \dots. \quad (1)$$

This iteration is quadratically convergent. If the initial values $z_i^{(0)}$ ($i = 1, \dots, n$) are sufficiently close to ζ_i , then it is known that the iterative sequence $\{z_i^{(k)}\}_{k=0}^{\infty}$ is well defined and convergent to the corresponding zero for $i = 1, \dots, n$.

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Let $r_i^{(k)} = \min_{j \neq i} |z_i^{(k)} - z_j^{(k)}|$, $\rho_i^{(k)} = \min_{j \neq i} |\zeta_i^{(k)} - \zeta_j^{(k)}|$, and

$$\begin{aligned} e_i^{(k)} &= |z_i^{(k)} - \zeta_i^{(k)}| / \rho_i^{(k)}, \\ \varepsilon_i^{(k)} &= |z_i^{(k)} - \zeta_i^{(k)}| / r_i^{(k)}, \\ \delta_i^{(k)} &= |z_i^{(k+1)} - z_i^{(k)}| / r_i^{(k)}. \end{aligned}$$

Denote $e^{(k)} = (e_1^{(k)}, \dots, e_n^{(k)})^T$, $\varepsilon^{(k)} = (\varepsilon_1^{(k)}, \dots, \varepsilon_n^{(k)})^T$ and $\delta^{(k)} = (\delta_1^{(k)}, \dots, \delta_n^{(k)})^T$. First we give the definitions of two criterions.

Definition 1. $\delta^{(0)}$ is called the first kind of criterion of Durand-Kerner method.

Definition 2. $e^{(0)}$ or $\varepsilon^{(0)}$ is called the second kind of criterion of an iteration.

From the definitions above we see that the first criterion varies for the different iteration, but it does not include the information about the zeros of $f(z)$. And we also see that the second kind one is not dependent on the iterations, so it can be used to compare the convergent conditions with other iteration.

In this paper we mainly discuss the l_p -norm convergent coditions of $e^{(0)}$, $\varepsilon^{(0)}$ and $\delta^{(0)}$ for Durand-Kerner method.

2. On the Second Kind of Criterion

For the convergence of discussing, we only analyse the iteration one step of (1) and simply denote $e_i^{(0)}$ and $e_i^{(1)}$ by e_i and e'_i . Other notations have the same meaning except the special statements.

From (1) we have

$$|z'_i - \zeta_i| = \left| \prod_{j \neq i}^n \left(\frac{z_i - \zeta_j}{z_i - z_j} \right) - 1 \right| |z_i - \zeta_i|,$$

and

$$\left| \prod_{j \neq i}^n \left(\frac{z_i - \zeta_j}{z_i - z_j} \right) \right| \leq \left| \prod_{j \neq i}^n \left(1 + \frac{z_j - \zeta_j}{\zeta_i - \zeta_j - (\zeta_i - z_i) - (z_j - \zeta_j)} \right) \right| \leq \prod_{j \neq i}^n \left(1 + \frac{e_j}{1 - e_i - e_j} \right). \tag{2}$$

Let $1 \leq p \leq \infty$ and $1/p + 1/q = 1$. By Hölder inequality it holds $e_i + e_j \leq 2^{1/q} \|e\|_p$, where $\|e\|_p = \left(\sum_{i=1}^n e_i^p \right)^{1/p}$. Thus

$$\begin{aligned} \left| \prod_{j \neq i}^n \left(\frac{z_i - \zeta_j}{z_i - z_j} \right) \right| &\leq \prod_{j \neq i}^n \left(1 + \frac{e_j}{1 - 2^{1/q} \|e\|_p} \right) \leq \left(1 + \frac{\sum_{j \neq i}^n e_j}{(n-1)(1 - 2^{1/q} \|e\|_p)} \right)^{n-1} \\ &\leq \left(1 + \frac{\|e\|_p}{(n-1)^{1/p} (1 - 2^{1/q} \|e\|_p)} \right)^{n-1}. \end{aligned} \tag{3}$$

Hence

$$|z'_i - \zeta_i| \leq \left(\left(1 + \frac{\|e\|_p}{(n-1)^{1/p} (1 - 2^{1/q} \|e\|_p)} \right)^{n-1} - 1 \right) |z_i - \zeta_i|$$