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WAVELET RATIONAL FILTERS AND REGULARITY ANALYSIS*

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Abstract

In this paper, we choose the trigonometric rational functions as wavelet filters and use them to derive various wavelets. Especially for a certain family of wavelets generated by the rational filters, the better smoothness results than Daubechies' are obtained.

Key words: Wavelet, Filter, Rational filter, Regularity.

1. Introduction

We denote by $\phi(x)$ a scaling function which satisfies

$$\phi(x) = \sum_{k\in Z} h_k \phi(2x-k) \quad (\text{Z is the integer set}). \tag{1}$$

The Fourier transform of equation (1) is

$$\hat{\phi} = H(\omega/2)\hat{\phi}(\omega/2) \tag{2}$$

where $\hat{\phi}(\omega)$ is the Fourier transform of $\phi(x)$ and

$$H(\omega) = \frac{1}{2}\sum_{k\in Z} h_k e^{-ik\omega}.$$

We call $H(\omega)$ a filter.It satisfies

$$H(0) = 1, \quad |H(\omega)|^2 + |H(\omega+\pi)|^2 = 1 \tag{3}$$

When the expansion coefficient sequence $\{h_k\}$ of $H(\omega)$ is given, the wavelets corresponding to the $H(\omega)$ can be derived. For the $H(\omega)$ which is a trigonometric polynomial (in this case, we call $H(\omega)$ a polynomial filter), Daubechies has given the methods generating wavelets as well as the estimates of regularity[1][2].

In this paper, we choose $H(\omega)$ to be a trigonometric rational function to generate wavelets and give relative methods and theorems. For I-type rational filters (see the

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second section),they include Daubechies' filters. And for II-type rational filters,they include B spline wavelet filters and have linear phases.Especially for a certain family of wavelets generated by the rational filters, the better smoothness results than Daubechies' are obtained.

2. Rational Filters

For a filter

$$H(\omega) = (\frac{1+e^{-i\omega}}{2})^N F(e^{-i\omega})$$

where $F(e^{-i\omega}) = \sum_{k\in Z} f_k e^{-ik\omega}$, Daubechies has given the conditions of existence of wavelets[1]:

$$\text{(I)} \qquad \sup_{\omega\in R} \mid F(e^{-i\omega}) \mid < 2^{N-1}$$

$$\text{(II)} \qquad \sum_{k\in Z} \mid f_k \mid\mid k \mid^{\epsilon} < \infty \quad \text{for a certain} \quad \epsilon > 0$$

On the basis of the two conditions, we will study how to construct wavelets by a rational filter.

Definition For a filter $H(\omega) = ((1+e^{-i\omega})/2)^N F(e^{-i\omega})$, when $F(z)$ is a rational function or the modulus of a rational function, we call $H(\omega)$ a rational filter.

Assume $P(z)$ and $Q(z)$ are relatively prime polynomials with real coefficients. Then $\mid P(e^{-i\omega})/Q(e^{-i\omega}) \mid$ is a rational function in $\cos\omega$. Riesz' lemma allow us to conclude that there is a real coefficient rational function $F(z)$ such that

$$\mid F(e^{-i\omega}) \mid^2 = \mid \frac{P(e^{-i\omega})}{Q(e^{-i\omega})} \mid . \tag{4}$$

Let

$$\mid F(e^{-i\omega}) \mid^2 = \frac{S(y)}{T(y)}, \qquad y = \cos^2\frac{\omega}{2} \tag{5}$$

where $S(y)$ and $T(y)$ are positive polynomials in the intervel [0,1]. For the given $S(y)$ and $T(y)$, the following two types of the rational filters can be determined by (5):

$$H_I(\omega) = (\frac{1+e^{-i\omega}}{2})^N F(e^{-i\omega}), \qquad H_{II}(\omega) = (\frac{1+e^{-i\omega}}{2})^N \mid F(e^{-i\omega}) \mid .$$

They are respectively called I-type rational filters and II-type rational filters. For II-type rational filters,$H_{II}(-\omega) = e^{iN\omega} H_{II}(\omega)$. This implies that II-type rational filters have linear phases.We have known that the function $F(e^{-i\omega})$s in the filters of orthogonal B spline wavelets are the moduli of the trigonometric rational functions[3].Therefore,the wavelets derived by II-type rational filters can include B spline wavelets.

For a I-type rational filter, we may use power series expansion to obtain sequence $\{h_k\}$. By the property of power series, we know that the condition (II) can be satisfied.