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DIRECT ITERATIVE METHODS FOR RANK DEFICIENT GENERALIZED LEAST SQUARES PROBLEMS*[1)][2)]

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Abstract

The generalized least squares (LS) problem

$$\min_{x\in\mathbf{R}^n}(Ax-b)^TW^{-1}(Ax-b)$$

appears in many application areas. Here W is an $m\times m$ symmetric positive definite matrix and A is an $m\times n$ matrix with $m\geq n$. Since the problem has many solutions in rank deficient case, some special preconditioned techniques are adapted to obtain the minimum 2-norm solution. A block SOR method and the preconditioned conjugate gradient (PCG) method are proposed here. Convergence and optimal relaxation parameter for the block SOR method are studied. An error bound for the PCG method is given. The comparison of these methods is investigated. Some remarks on the implementation of the methods and the operation cost are given as well.

Key words: Rank deficient generalized LS problem, block SOR method, PCG method, convergence, optimal parameter

1. Introduction

The generalized LS problem

$$\min_{x\in\mathbf{R}^n}(Ax-b)^TW^{-1}(Ax-b) \tag{1.1}$$

is frequently found in solving problems from statistics, engineering, economics, image and signal processing. Here $A\in\mathbf{R}^{m\times n}$ with $m\geq n$, $b\in\mathbf{R}^m$ and $W\in\mathbf{R}^{m\times m}$ is

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symmetric positive definite. The large sparse rank deficient generalized LS problems appear in computational genetics when we consider mixed linear model for tree or animal genetics [2], [3], [5].

Recently, Yuan [9] and [10], Yuan and Iusem [11] considered direct iterative methods for the problem (1.1) by preconditioned techniques when A has full column rank. They proposed the block SOR-type method and the PCG method for solving the problem (1.1). They also showed that the PCG method is better than the block SOR-type method. However there are few papers to deal with the iterative methods for solving the rank deficient generalized LS problems. It motivates us to propose two iterative methods, the block SOR method and the PCG method, for solving the rank deficient generalized LS problems.

In order to speed up the convergence rate of the block SOR method and the PCG method, the key factor is to find a good preconditioner. Several algorithms were proposed recently in [1], [6] and [7] to find the preconditioners for the LS problems. By using those algorithms, we can develop some good preconditioners of our methods for solving the rank deficient generalized LS problems.

The outline of the paper is as follows. In Section 2, an augmented system for the rank deficient generalized LS problem is given with special transformation. The new system is the base of all work done in this paper. A block SOR method for the problem (1.1) is studied in Section 3. The PCG method is established in Section 4. Comparison of these methods and some remarks on the methods are given in the last section. We always assume that $\text{rank}(A) = k < n$ and W is symmetric positive definite in this paper.

2. Preconditioned Systems

We suppose that A has the following partition

$$A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \tag{2.1}$$

where A_1 is a $k \times n$ full row rank matrix, i.e., $\text{rank}(A_1) = k = \text{rank}(A)$, and A_2 is an $(m-k) \times n$ matrix.

Lemma 2.1[4]. *We have*

$$\mathbf{R}(A^T) = \mathbf{R}(A_1^T) \quad \text{and} \quad \mathbf{N}(A) = \mathbf{N}(A_1)$$

where $\mathbf{R}(B)$ *is the range of the matrix* B *and* $\mathbf{N}(B)$ *is the null space of* B.

For the rank deficient LS problems, since there are many solutions in this case, we are interested in the minimum 2-norm solution which appears in some applications. It