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THE L^2- NORM ERROR ESTIMATE OF NONCONFORMING FINITE ELEMENT METHOD FOR THE 2ND ORDER ELLIPTIC PROBLEM WITH THE LOWEST REGULARITY [*1)]

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Abstract

The abstract L^2-norm error estimate of nonconforming finite element method is established. The uniformly L^2-norm error estimate is obtained for the nonconforming finite element method for the second order elliptic problem with the lowest regularity, i.e., in the case that the solution $u \in H^1(\Omega)$ only. It is also shown that the L^2-norm error bound we obtained is one order heigher than the energe-norm error bound.

Key words: L^2-norm error estimate, nonconforming f.e.m., lowest regularity

1. Introduction

This paper is concerned with the uniformly L^2- norm error estimate of the nonconforming finite method for the second order elliptic problem with the lowest regularity, i.e., in the case that the solution $u \in H^1(\Omega)$ only, but not in $H^2(\Omega)$.

For the conforming finite element method of the second order elliptic problem, it is well known that using the Aubin-Nitsche lemma obtained the L^2- norm error bound, which is one order of h, the parameter of triangulation, higher than the H^1- norm error bound, in the case that the solution u of the primale problem is smooth enough, i.e., $u \in H^2(\Omega)$ (c.f.[1]). And recently, Schatz and Wang [2] considered the uniformly L^2- norm error bound for the conforming finite element method of second order elliptic problem in the case that the solution u is not smooth enough, i.e., $u \in H^1(\Omega)$ only, but not in $H^2(\Omega)$.

In order to consider the L^2- norm error estimate for the nonconforming finite element method, we need the Aubin-Nitsche lemma for the nonconforming finite element method, which has been considered in [4], and for which we now give a clear expression and a simple proof.

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Firstly, let us state the Aubin-Nitsche lemma for the conforming finite element method.

Consider the variational elliptic problem as follows

$$\begin{cases} \text{Find } \ u \in H_0^1(\Omega) \ \text{ such that} \\ a(u,v) = (f,v) \qquad \forall v \in H_0^1(\Omega), \end{cases} \tag{1.1}$$

where

$$a(u,v) \equiv \int_\Omega a_{ij}(x)\partial_i u \partial_j v dx, \tag{1.2}$$

$$(f,v) \equiv \int_\Omega f \cdot v dx \tag{1.3}$$

and $a_{ij}(x) \in L^\infty(\Omega), f \in L^2(\Omega)$,

$$\sum_{i,j=1}^{2} a_{ij}(x)\xi_i\xi_j \geq \alpha \sum_{i=1}^{2} \xi_i^2 \quad \forall x \in \Omega, \quad \xi = (\xi_1, \xi_2)^T \in R^2. \tag{1.4}$$

Then the conforming finite element approximation of (1.1) is as follows, let $\tilde{V_h} \subset H_0^1(\Omega)$ be the finite element subspace of $H_0^1(\Omega)$

$$\begin{cases} \text{Find } \ u_h \in \tilde{V_h}, \qquad \text{such that} \\ a(u_h, v_h) = (f, v_h) \qquad \forall \ \ v_h \in \tilde{V_h}. \end{cases} \tag{1.5}$$

Then it is well known that

Theorem 1. (Aubin-Nitsche Lemma)(c.f.[1])

Let u and u_h be the solutions of the problems (1.1) *and* (1.5) *respectively, then there exists $C =$ Const.> 0, such that*

$$\|u - u_h\|_0 \leq \|u - u_h\|_1 \sup_{g \in L^2(\Omega)} \{ \frac{1}{\|g\|_0} \inf_{\phi_h \in \tilde{V_h}} \|\phi_g - \phi_h\|_1 \}, \tag{1.6}$$

where, for any given $g \in L^2(\Omega), \phi_g \in H_0^1(\Omega)$ such that

$$a(v, \phi_g) = (g, v) \qquad v \in H_0^1(\Omega). \tag{1.7}$$

Corollary 2. ([2])

Assume that $f \in L^2(\Omega)$, then given any $\epsilon > 0$, there exists an $h_0 = h_0(\epsilon) > 0$ such that for all $0 < h \leq h_0(\epsilon)$,

$$\|u - u_h\|_0 \leq \epsilon \|u - u_h\|_1. \tag{1.8}$$

The proof can be completed from that $\|\phi_g - (\phi_g)_h\|_1 \leq \epsilon \|g\|_0$ (c.f.[2]) and (1.6).

Note that the Corollary 2 shows that the L^2- norm error bound is one order of ϵ higher than the H^1- norm error bound for the conforming finite element approximation to the second order problem in the case that the solution $u \in H^1(\Omega)$ only, but not in $H^2(\Omega)$.