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FOURIER-LEGENDRE PSEUDOSPECTRAL METHOD FOR THE NAVIER-STOKES EQUATIONS[*1)]

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Abstract

In this paper, we construct a Fourier-Legendre pseudospectral scheme for the unsteady Navier-Stokes equations. This method easily deals with nonlinear terms and saves computational time. The strict error estimations are given.

Key Words: Navier-Stokes equations, Fourier-Legendre pseudospectral method, error estimation.

1. Introduction

The mixed spectral and pseudospectral methods are successful to numerically solve the semi-periodic problems of incompressible fluid flows (see [1-6]). This paper is devoted to the Fourier-Legendre pseudospectral method for the two-dimensional unsteady Navier-Stokes equations with semi-periodic boundary condition. This method is performed easily and has the same high accuracy as spectral method has.

Let $x = (x_1, x_2)^T$ and $\Omega = I_1 \times I_2$ where $I_1 = \{x_1 / -1 < x_1 < 1\}$, $\quad I_2 = \{x_2 / -\pi < x_2 < \pi\}$. We denote by $U(x,t)$ and $P(x,t)$ the speed and the pressure. Let $\partial_t = \frac{\partial}{\partial t}$ and $\partial_j = \frac{\partial}{\partial x_j} \quad (j = 1, 2)$. We consider the Navier-Stokes equations as follows

$$\begin{cases} \partial_t U + (U \cdot \nabla) U - \nu \nabla^2 U + \nabla P = f, & \text{in} \quad \Omega \times (0,T], \\ \nabla \cdot U = 0, \qquad \text{in} \quad \Omega \times (0,T], \\ U(x,0) = U_0(x), \quad P(x,0) = P_0(x), \quad \text{in} \quad \Omega, \end{cases} \tag{1.1}$$

where $\nu > 0$ is the kinetic viscosity, $U_0(x)$ and $P_0(x)$ are the initial values. Assume that all functions in (1.1) have the period 2π for x_2. We also suppose that U satisfies the homogeneous boundary conditions in the x_1-direction

$$U(-1, x_2, t) = U(1, x_2, t) = 0, \quad \forall x_2 \in I_2.$$

Besides, to fix $P(x,t)$, we require

$$\mu(P) \equiv \int_\Omega P(x,t) dx = 0, \quad \forall t \in [0,T].$$

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We denote by $(\cdot,\cdot)$ and $\|\cdot\|$ the usual inner product and norm of $L^2(\Omega)$, etc.. Let $C^\infty_{0,p}(\Omega)$ be the subset of $C^\infty(\Omega)$, whose elements vanish at $x_1=\pm 1$ and have the period 2π for $x_2\in I_2$. $H^1_{0,p}(\Omega)$ is the closure of $C^\infty_{0,p}(\Omega)$ in $H^1(\Omega)$.

2. The Scheme

Let M and N be positive integers. Assume that there exist positive constants c_1 and c_2 such that

$$c_1N\le M\le c_2N.$$

We denote by $\mathcal{P}_M$ the space of all polynomials with degree $\le M$, defined on I_1. Let

$$V_M=\{v(x_1)\in\mathcal{P}_M/v(-1)=v(1)=0\}.$$

Set l be integer, and

$$\tilde{V}_N=\mathrm{Span}\{e^{ilx_2}/\,|l|\le N\}.$$

Let V_N be the subset of $\tilde{V}_N$, containing all real-valued functions. Define

$$V_{M,N}=(V_M\times V_N)^2,\quad S_{M-1,N}=\{v\in\mathcal{P}_{M-1}\times V_N/\mu(v)=0\}.$$

Let $P^1_{M,N}:(H^1_{0,p}(\Omega))^2\longrightarrow V_{M,N}$ be the projection operator such that for any $u\in(H^1_{0,p}(\Omega))^2$,

$$(\nabla(u-P^1_{M,N}u),\nabla v)=0,\quad\forall v\in V_{M,N}.$$

While $P_{M-1,N}:L^2(\Omega)\longrightarrow\mathcal{P}_{M-1}(I_1)\times V_N$ is the orthogonal projection such that for any $u\in L^2(\Omega)$,

$$(u-P_{M-1,N}u,v)=0,\quad\forall v\in\mathcal{P}_{M-1}\times V_N.$$

Obviously, if $u\in L^2(\Omega)$ and $\mu(u)=0$, then $\mu(P_{M-1,N}u)=0$.

Now, let $\{x_1^{(j)},\omega^{(j)}\}$ be the nodes and weights of Gauss-Lobatto integration, i.e.,

$$x_1^{(0)}=-1,x_1^{(M)}=1,x_1^{(j)}(j=1,\cdots,M-1)\text{ zeroes of }L'_M,$$
$$\omega^{(j)}=\frac{2}{M(M+1)(L_M(x_1^{(j)}))^2},j=0,\cdots,M,$$

where L_M is the Legendre polynomial of degree M. Let $h=\frac{2\pi}{2N+1}$ be the mesh size for x_2. Define

$$\Omega_{M,N}=\{(x_1^{(j)},lh)/1\le j\le M-1,-N\le l\le N\},$$
$$\bar{\Omega}_{M,N}=\{(x_1^{(j)},lh)/0\le j\le M,-N\le l\le N\}.$$

The discrete inner products and norms are defined as follows

$$<u,v>_M=\sum_{j=0}^{M}u(x_1^{(j)})v(x_1^{(j)})\omega^{(j)},$$

$$(u,v)_{M,N}=\frac{1}{2N+1}\sum_{j=0}^{M}\sum_{l=-N}^{N}u(x_1^{(j)},lh)\bar{v}(x_1^{(j)},lh)\omega^{(j)},$$

$$\|u\|_{M,N}=(u,u)^{\frac12}_{M,N},\quad|u|_{1,M,N}=(\sum_{j=1}^{2}\|\partial_ju\|^2_{M,N})^{\frac12}.$$