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D-CONVERGENCE AND STABILITY OF A CLASS OF LINEAR MULTISTEP METHODS FOR NONLINEAR DDES*1)

Cheng-jian Zhang Xiao-xin Liao
(*Department of Mathematics, Huazhong University of Science and Technology, Wuhan 430074, China*)

Abstract

This paper deals with the error behaviour and the stability analysis of a class of linear multistep methods with the Lagrangian interpolation (LMLMs) as applied to the nonlinear delay differential equations (DDEs). It is shown that a LMLM is generally stable with respect to the problem of class $D_{\sigma,\gamma}$, and a p-order linear multistep method together with a q-order Lagrangian interpolation leads to a D-convergent LMLM of order $\min\{p, q+1\}$.

Key words: D-Convergence, Stability, Multistep methods, Nonlinear DDEs.

1. Introduction

Consider the following nonlinear delay problem

$$\begin{cases} y'(t) = f(t, y(t), y(t-\tau)), & t \in [t_0, T], \quad (1.1a) \\ y(t) = \varphi(t), & t \in [t_0 - \tau, t_0], \quad (1.1b) \end{cases}$$

where $y : R \to C^N, \tau > 0$ is a delay term, $f : [t_o, T] \times C^N \times C^N \to C^N$ and $\varphi(t) : [t_0 - \tau, t_0] \to C^N$ denotes a given initial function. Thoroughout this paper , the problem (1.1) is supposed to have a unique solution y(t), which satisfies

$$\| y^{(i)}(t) \| \leq M_i, \quad t \in [t_0 - \tau, T]$$

here norm $\| \bullet \|$ is defined by $\| x \|^2 = < x, x >$ $(\forall x \in C^N)$, and $M_i > 0$ are some constants.

Definition 1.1.[1] *The class of all delay problems of the form (1.1) with*

$$\begin{cases} Re < u - v, f(t, u, \tilde{u}) - f(t, v, \tilde{u}) > \leq \sigma \| u - v \|^2 & (1.2) \\ \| f(t, u, \tilde{u}) - f(t, u, \tilde{v} \| \leq \gamma \| \tilde{u} - \tilde{v} \|, & (1.3) \\ where\ t \in [t_o, T], u, \tilde{u}, v, \tilde{v} \in C^N,\ and\ constants\ \sigma,\ \gamma\ satisfy \\ 0 \leq \gamma \leq -\sigma \end{cases}$$

is denoted by $D_{\sigma,\gamma}$.

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The following proposition on stability of the problem (1.1) can be inferrded directly by a result of L. Torelli [1].

Proposition 1.1. *Suppose the problem* (1.1) *belongs to the class* $D_{\sigma,\gamma}$. *Then for any two solutins* $y(t)$ *and* $z(t)$ *of the equation* (1.1a) *we have*

$$\| y(t) - z(t) \| \leq \max_{x \in [t_0 - \tau, t_0]} \| \varphi(x) - \psi(x) \|,$$

where $\varphi(t)$ *and* $\psi(t)$ *are the two initial functions corresponding to the solutions* $y(t), z(t)$.

Moreover, it is remarkable that H.J.Tian and J.X.Kuang [2] gave a Theorem on asymptotic stability of (1.1) with an adaptation to the conditions (1.2)–(1.3).

So far, a lot of results on nonlinear stability and convergence of the numerical solutions of DDEs have been obtained ($cf.$[$1-7$]). However, these results were achived under the classical Lipschitz condition except those of the paper [$1,6,7$], which deal only with Runge-Kutta methods. In view of what aboves, we study convergence and stability of a class of variable-coefficient LMLMs for the problem of class $D_{\sigma,\gamma}$. and present some significant results in this paper.

2. The Methods and the Basic Lemmas

Consider variable-coefficient LMLMs ($cf.$[8]) for (1.1)

$$\sum_{i=0}^{k} \alpha_i [y_{n+i} - h\beta_i f(t_{n+i}, y_{n+i}, y^h(t_{n+i} - \tau))] = 0, \tag{2.1}$$

where k is a positive integer; $n = 0, 1, 2, \ldots, N$, and $(N+k)h \leq T - t_0, h > 0$ is a stepsize independent of n; the coefficients α_i, β_i are real-valued functions of h and there exists a constant $h_1 > 0$ such that for $h \in (0, h_1]$,

$$\alpha_k = 1, \quad \sum_{i=0}^{k} \alpha_i = 0, \quad \max_{i \in I_0} \alpha_i \leq 0, \quad \max_{i \in I_0} | \beta_i | \leq \beta_k < \beta, \tag{2.2}$$

where $I_0 = \{0, 1, 2, \ldots, k-1\}, \beta > 0$ is a constant; $y_{n+i}, y^h(t_{n+i} - \tau) \in C^N$ are approximations to $y(t_{n+i})$ and $y(t_{n+i} - \tau)$ respectively, and $y^h(\bullet)$ is determined by Lagrangian interpolation

$$y^h(t_m + \delta h) = \begin{cases} \displaystyle\sum_{j=-r}^{s} L_j(\delta) y_{m+j}, t_0 < t_m + \delta h \leq T, \\ \varphi(t_m + \delta h), t_0 - \tau \leq t_m + \delta h \leq t_0, \end{cases} \tag{2.3}$$

where $\delta \in [0, 1), r, s$ are positive integers, $t_m = t_0 + mh$ (m denotes a integer) and

$$L_j(\delta) = \prod_{\substack{l=-r \\ l \neq j}}^{s} \left(\frac{\delta - l}{j - l}\right)$$