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# ON CONVERGENCE OF NOUREIN ITERATIONS FOR SIMULTANEOUS FINDING ALL ZEROS OF A POLYNOMIAL*1)

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**Abstract**

In this paper, we give the first estimation conditions for Nourein iterations for simultaneous finding all zeros of a polynomial under which the iteration processes are guaranteed to converge.

*Key words*: Polynomial zeros, Parallel iteration, Nourein iterations, Point estimation, Convergence.

## 1. Introduction

Suppose that

$$f(t) = \sum_{i=0}^{n} a_i t^{n-i} = \prod_{j=1}^{n} (t - \xi_i), \quad a_0 = 1 \tag{1}$$

is a monic polynomial of degree $n$ with complex coefficients. Some authors have studied the parallel iterations without derivatives for simultaneous finding all zeros $\xi_1, \xi_2, \cdots, \xi_n$ of $f(t)$ (see [1]-[10],[13], [14], [16]). The famous one is Durand-Kerner iteration with the form

$$x_i^{k+1} = x_i^k - u_i^k \quad i = 1, 2, \cdots, n, \quad k = 0, 1, \cdots, \tag{2}$$

where $x_i^k$ is the k-th approximation of $\xi_i (1 \le i \le n)$ and

$$u_i^k = \frac{f(x_i^k)}{\prod\limits_{j \ne i} (x_i^k - x_j^k)}, \quad i = 1, \cdots, n, \quad k = 0, 1, \cdots, \tag{3}$$

which does not require any information about the derivatives (see[3], [4], [6], [10], [14]). However, formula (2) appeared for the first time in Weierstrass' work [13],p.258] connected with a constructive proof of the fundamental theorem of algebra. The convergence of (2) is quadratic if $\xi_i \ne \xi_j$ for $i \ne j$, which was first proved by K. Dochev [3] and later by other authors.

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The other two iterations without derivatives are of the form

$$x_i^{k+1} = x_i^k - \frac{u_i^k}{1 + \sum\limits_{j \neq i} \frac{u_j^k}{x_i^k - x_j^k}} \qquad i = 1, 2, \cdots, n, \quad k = 0, 1, \cdots, \tag{4}$$

and

$$x_i^{k+1} = x_i^k - \frac{u_i^k}{1 + \sum\limits_{j \neq i} \frac{u_j^k}{x_i^k - u_i^k - x_j^k}} \qquad i = 1, 2, \cdots, n, \quad k = 0, 1, \cdots. \tag{5}$$

The iteration formula (4) was derived by Börsch-Supan [1], and later by Nourein [7], and (5) by Nourein [8]. As in Carstensen & Petkovic [2], (4) and (5) are both called Nourein iterations, and the order of convergence of them is three and four, respectively, if $\xi_i \neq \xi_j$ for $i \neq j$.

The concept of "point estimation", which gives the convergence criteria for iterations from data at initial points, was first proposed by S. Smale[11], and has attracted many authors (see, for example, [10], [12], [15] and references therein). In [16], the first author of this paper gave the conditions of the initials for the Durand-Kerner iteration under which the iteration (2) is guaranteed to converge to the zeros of $f(t)$, which is actually a point estimation convergence criterion.

In this paper we consider the point estimation for Nourein iterations (4) and (5). Our Theorems are different from that in [2], where a local convergence theorem is given, in which the conditions of convergence are concerning with the zeros of the polynomial. However, the zeros are unknown in advance. Therefore, the conditions are unable to verify. But all conditions in the following theorems depend only on the initials and can be verified.

## 2. Main Results and A Numerical Example

For purposes of brevity, all formulas, sums and products (such as in (2), (3), (4) and (5) above) involving indices $i, j$ and $\nu$ will assume the range $1, 2, \cdots, n$ and the iterative index $k = 0, 1, \cdots$, unless explicitly stated otherwise. Throughout this paper we will assume that $n \geq 3$.

Let

$$\delta_k = \max_{1 \leq i \leq n} |u_i^k|, B_k = \max_{j \neq i} |x_i^k - x_j^k|^{-1}, s_k = B_k \delta_k, \epsilon_n = \frac{1}{2(n+1)},$$

$$\phi_1(s) = \frac{1}{1-(n-1)s}, \qquad \phi_2(s) = \frac{1-s}{1-ns},$$

$$g_1(s) = \frac{(n-1)s^2}{[1-(n+1)s]^2} [1 + \frac{s}{1-(n+1)s}]^{n-2},$$

$$g_2(s) = \frac{(n-1)^2 s^3}{[1-(n+2)s+2s^2]^2} [1 + \frac{s(1-s)}{1-(n+2)s+2s^2}]^{n-2}.$$