Journal of Computational Mathematics, Vol.18, No.1, 2000, 25–42.

WAVELET METHOD FOR BOUNDARY INTEGRAL EQUATIONS*

Pin-wen Zhang
(*School of Mathematical Sciences, Peking University, Beijing 100871, China*)

Yu Zhang
(*Institute of Computational Mathematics and Scientific/Engineering Computing, Chinese Academy of Sciences, Beijing 100080, China*)

Abstract

In this paper, we show how to use wavelet to discretize the boundary integral equations which are both singular and ill-conditioned. By using an explicit diagonal preconditioning, the condition number of the corresponding matrix is bounded by a constant, while the sparse structure speed up the iterative solving process. Using an iterative method, one thus obtains a fast numerical algorithm to solve the boundary integral equations.

Key words: Wavelet bases, Boundary integral equation, Preconditioning.

1. Introduction

The application of wavelets to signal and image processing has been successful. But there are few results about numerical solution for the partial differential equations. We think that it is enough only to use the coefficients $\{h_n\}$ (as a filter) in signal and image processing, but it is not sufficient for numerical computation[7][8][10], where we consider wavelet as a special function series instead of a filter. Then it brings some problems such as the complication of function value computation and the difficulty to handle the boundary conditions[8][13], etc. For the first problem, we have solved in [16]. For the second problem, some researchers consider the periodic problems to escape the complicated boundary [3], others consider the boundary integral problems. In [3], the author discussed the potential integral equation of the 2D Laplace operator, as we know, the kernel is no-singular and well-conditioned.

In this paper we examine the feasibility of applying wavelet based numerical methods to solve elliptic equations. We use compactly supported wavelets and develop a wavelet boundary element method to handle the boundary conditions. The boundary element method has been firmly established as an important alternative technique to the prevailing numerical methods of analysis in continuum mechanics[14][17], and many others which can be written as a function of a potential and whose governing equation is the classical Laplace or Poisson equation. It reduces the problem's dimension and can be used when the domain is infinite. In contrast to the finite difference methods

* Received June 26, 1996.

and finite element methods, the classical discretization of the boundary integral usually has slow decay away from the diagonal and leads to a large non-sparse linear system. As we know, directly applying a dense matrix to a vector requires roughly N^2 operations. Therefore, the efficiency induced by lowering the dimensions is distroyed by the cost brought by the dense matrices. This problem can be solved by the wavelet method proposed by Beylkin, Coifman and Rokhlin in [1]. According to the framework of Zygmund-Calderon operators studied extensively in [1], if we project the integral operator in a wavelet basis, since the basis function satisfies the vanishing moment condition, the coefficients away from the diagonal will be small. Neglecting these small entries yields a finger-like sparse linear system.

In this paper we mainly discuss two kinds of integral induction methods: single-layer induction and natural boundary induction. Both of them are ill-conditioned and have singularity in their kernels. In order to avoid these problems, we construct numerical algorithm to avoid the singularity, and find an easily inheritable matrix D such that $D^{-1}MD^{-1}$ will have a better condition number κ. We prove that using the wavelet basis, a diagonal matrix D yields $\kappa = O(1)$.

2. Boundary Integral Equations

Let Ω be a domain in the plane $\mathbf{R}^2$ and Γ its boundary, we discuss the Laplace equation:

$$\triangle u = 0, \qquad x \in \Omega, \tag{2.1}$$

with boundary conditions of Dirichlet type

$$u(x) = f(x) \qquad \text{on } \Gamma, \tag{2.2}$$

or the Neumann type

$$\frac{\partial u(x)}{\partial n} = g(x) \qquad \text{on } \Gamma, \tag{2.3}$$

where n is the unit outward normal to surface Γ.

The Neumann problem (2.1) and (2.3) has a solution only if the consistency condition

$$\int_\Gamma g(x) dS_x = 0 \tag{2.4}$$

holds and this solution is unique only to within an arbitrary additive constant.

It is well known that the above elliptic boundary-value problem can be reduced into several kinds of integral equations. To obtain an integral equation for the solution of the Dirichlet problem, the classical approach is to assume that the unknown function u may be expressed solely as a single-layer potential with unknown density σ,

$$u(x) = \int_\Gamma \sigma(y) \ln \frac{1}{|x-y|} dS_y + C, \qquad x \in \Omega, \tag{2.5}$$

where C is a constant which will be determined later. Since the kernel in this equation is continuous as x passes through the surface, the limit of equation (2.5) as x is taken