Journal of Computational Mathematics, Vol.17, No.6, 1999, 639–644.

ORDER RESULTS FOR ALGEBRAICALLY STABLE MONO-IMPLICIT RUNGE-KUTTA METHODS*[1)]

Ai-guo Xiao

(*1. Department of Mathematics, Xiangtan University, Xiangtan 411105, China*
2. ICMSEC, Chinese Academy of Sciences, Beijing 10080, China)

Abstract

It is well known that mono-implicit Runge-Kutta methods have been applied in the efficient numerical solution of initial or boundary value problems of ordinary differential equations. Burrage(1994) has shown that the order of an s-stage mono-implicit Runge-Kutta method is at most s+1 and the stage order is at most 3. In this paper, it is shown that the order of an s-stage mono-implicit Runge-Kutta method being algebraically stable is at most min$(\tilde{s}, 4)$, and the stage order together with the optimal B-convergence order is at most min(s,2), where

$$\tilde{s} = \begin{cases} s+1 & if \quad s = 1, 2, \\ s & if \quad s \geq 3. \end{cases}$$

Key words: Ordinary differential equations, Mono-implicit Runge-Kutta methods, Order, Algebraical stability.

1. Introduction

Consider the initial value problem

$$\begin{cases} y'(t) &= f(t, y(t)) \quad t \geq 0, \qquad f: [0, +\infty) \times R^N \rightarrow R^N, \\ y(0) &= y_0 \in R^N \end{cases} \tag{1.1}$$

which is assumed to have a unique solution $y(t)$ on the interval $[0, +\infty)$.

For solving (1.1), consider the s-stage implicit Runge-Kutta (IRK) method

$$\begin{cases} y_{n+1} &= y_n + h \sum_{i=1}^{s} b_i f(t_n + c_i h, Y_i) \\ Y_i &= y_n + h \sum_{j=1}^{s} a_{ij} f(t_n + c_j h, Y_j), \qquad 1 \leq i \leq s \end{cases} \tag{1.2}$$

and the s-stage mono-implicit Runge-Kutta (MIRK) method[2,5]

* Received April 16, 1996.

1) This work was supported by National Natural Science Foundation Of China.

$$\begin{cases} y_{n+1} &= y_n + h\sum\limits_{i=1}^{s} b_i f(t_n + c_i h, Y_i) \\ Y_i &= (1-\nu_i)y_n + \nu_i y_{n+1} + h\sum\limits_{j=1}^{i-1} x_{ij} f(t_n + c_j h, Y_j), \quad 1\leq i\leq s \end{cases} \tag{1.3}$$

where $h\rangle 0$ is the stepsize, b_i, c_i, ν_i, x_{ij} and a_{ij} are real constants, $b_i\neq 0$, $\sum\limits_{i=1}^{s} b_i = 1, c_i\neq c_j$ when $i\neq j$, Y_i and y_n approximate $y(t_n + c_i h)$ and $y(t_n)$ respectively, $t_n = nh$ $(n\geq 0)$. The methods (1.2) and (1.3) can be given in the tableau forms respectively:

$$\begin{array}{c|c} \mathrm{c} & \mathrm{A} \\ \hline & b^T \end{array} \tag{1.4}$$

and

$$\begin{array}{c|c|c} \mathrm{c} & \nu & \mathrm{X} \\ \hline & & b^T \end{array} \tag{1.5}$$

where $c = (c_1, c_2, \cdots, c_s)^T, b = (b_1, b_2, \cdots, b_s)^T, \nu = (\nu_1, \nu_2, \cdots, \nu_s)^T, A = [a_{ij}]$ is an $s\times s$ matrix, $X = [x_{ij}]$ is an $s\times s$ matrix with $x_{ij} = 0$, when $i\leq j$. The method (1.5) is equivalent to the IRK method (1.4) with the coefficient matrix $A = X + \nu b^T$. The method (1.4) is said to be algebraically stable[4,7], if the matrixes $M = BA + A^T B - bb^T$ and $B = diag(b)$ are nonnegative definite.

A number of interesting subclasses of the IRK methods have recently been identified and investigated in the references. These methods represent attempts to trade-off the higher accuracy of the IRK methods for methods which can be implemented more efficiently. These methods include singly-implicit Runge-kutta (SIRK) methods[1,6,7], diagonally implicit Runge-Kutta (DIRK) methods[1,6,7],and MIRK methods[2,5]. Burrage[5] has shown that the order of an s-stage MIRK method is at most s+1 and the stage order is at most 3. In this paper, it is shown that the order of an s-stage MIRK method being algebraically stable is at most min($\tilde{s}$,4) and the stage order together with the optimal B-convergence order is at most min(s,2), here and in the following sections,

$$\tilde{s} = \begin{cases} s+1 & if \quad s = 1, 2, \\ s & if \quad s\geq 3. \end{cases}$$

2. Main Results and Proofs

For the method (1.4) or (1.5), we introduce the simplifying conditions[1,7]:

$$\begin{array}{lll} B(p): & b^T c^{k-1} = \frac{1}{k}, & k = 1, 2, \cdots, p \\ C(p): & Ac^{k-1} = \frac{c^k}{k}, & k = 1, 2, \cdots, p \\ D(p): & b^T C^{k-1} A = \frac{b^T - b^T C^k}{k}, & k = 1, 2, \cdots, p \end{array}$$

where $c^k = (c_1^k, c_2^k, \cdots, c_s^k)^T, C^k = diag(c^k)$. $\max\{p : B(p)$ and $C(p)$ hold at the same time$\}$ is said to be the stage order of the method (1.4). Since the MIRK method (1.5)