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ON RAYLEIGH QUOTIENT MATRICES: THEORY AND APPLICATIONS*

Xin-guo Liu
(*Department of Applied Mathematics Ocean University of Qingdao, Qingdao 266003, China*)

Abstract

Many authors have studied the Rayleigh quotient and Rayleigh quotient matrix. This paper consists of two parts. First, generalizations of some results on the Rayleigh quotient are proved. Second, we give some applications of these theoretical results.

Key words: Rayleigh quotient Matrix, Eigenvalue, Approximation.

1. Introduction

Throughout this paper we shall use the following notation. $R^{m\times n}$ and $C^{m\times n}$ denote the sets of real and complex $m\times n$ matrices, respectively, R^n and C^n denote the sets of real and complex n-dimentional column vectors, respectively. The superscript H means the conjugate transpose of matrix. I_n is the $n\times n$ identity matrix, and 0 is the null matrix. $R(A)$ stands for the column space of a matrix A; $\lambda(A)$ denotes the set of the eigenvalues of matrix A. $\lambda(A,B)$ denotes the set of the generalized eigenvalues of a regular matrix-pair $\{A,B\}$. $\sigma(A)$ the set of the singular values of matrix A.$\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ denote the smallest and largest eigenvalue of Hermitian matrix A, respectively. $\sigma_{\min}(A)$ is the smallest singular value of matrix A. $\|\ \|$ refers to a uniformly generalized, unitarily invariant norm for matrices. $\|\ \|_2$ denotes the Euclidean norm for vectors and spectral norm for matrices, respectively. $\|\ \|_F$ is the Frobenius norm. For X_1, $Y_1\in C^{m\times p}$ with $X_1^HX_1=Y_1^HY_1=I_p$, the matrix $\theta(R(X_1),R(Y_1))$ is defined by

$$\theta(R(X_1),R(Y_1))=\arccos(X_1^HY_1Y_1^HX_1)^{1/2}\geq 0$$

Let $A\in C^{n\times n}$ be a Hermitian matrix, and $Y_1\in C^{n\times p}$ satisfy $Y_1^HY_1=I_p$. Then the matrix $H_1=Y_1^HAY_1$ is called the Rayleigh quotient matrix of A with respect to Y_1. If $p=1$, then $y_1^HAy_1$ is called the Rayleigh quotient of A respect to y_1.

First of all we cite some important results on the Rayleigh quotient. let A be $n\times n$ Hermitian matrix, and $\lambda(A)=\{\lambda_j\}_{j=1}^n$, moreover, let $y_1\in C^n$ with $\|y_1\|_2=1$, and let

$$Ax_1=\lambda_1X_1,\|X_1\|_2=1,\quad X_1\in C^n$$
$$\mu_1=y_1^HAy_1,r=Ay_1-y_1\mu_1$$

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$$\theta = \arccos|y_1^H X_1|, \quad 0 \le \theta \le \pi/2$$
$$\delta = \min_{2\le j\le n} |\lambda_j - \mu_1|, \quad \Delta = \max |\lambda_j - \mu_1|$$
$$d = \min_{2\le j\le n} |\lambda_j - \lambda_1|, \quad D = \max_{2\le j\le n} |\lambda_j - \lambda_1|, \quad 2 \le j \le n.$$

Some elementary results are given in the following theorem, which delineates the most important relations between $\sin\theta$, $\|r\|_2$ and $\lambda_1 - \mu_1$.

Theorem[11]**.**

$$\sin\theta \le \|r\|_2/\delta \quad (\text{if } \delta > 0) \tag{1.1}$$
$$\|r\|_2 \le \frac{\Delta \sin\theta}{\sqrt{1-\sin^2\theta}} \quad (\text{if } \sin\theta < 1) \tag{1.2}$$
$$|\lambda_1 - \mu_1| \le \|r\|_2^2/\delta \quad (\text{if} \delta > 0, |\lambda_1 - \mu_1| < |\lambda_j - \mu_1|), \tag{1.3}$$
$$|\lambda_1 - \mu_1| \le D \sin^2\theta \tag{1.4}$$
$$|\lambda_1 - \mu_1| \le \frac{\|r\|_2 \sin\theta}{\sqrt{1-\sin^2\theta}} \quad (\text{if} \sin\theta < 1) \tag{1.5}$$

The inequalities (1.1)–(1.5) *have been extended to the case $p > 1$ by Sun*[10,11]*, Li*[5]*, Liu & Xu*[6]*, and Liu*[7]*. In this paper, we shall give some further generalizations of the inequalities* (1.1)–(1.5) *and applications of these theoretical results.*

2. Generalizations of the Rayleigh Quotient Matrix Theory

In this section, some extentions of the inequalities (1.1)–(1.5) are given. We shall study the eigenproblem, generalized eigenvalue problem and singular value problem.

2.1. Eigenproblem: $p = 1$

Let $A \in C^{n\times n}$, $y_1 \in C^n$ with $\|y_1\|_2 = 1$, and let

$$\mu_1 = y_1^H A y_1, \quad r = Ay_1 - y_1 u_1, \quad r_0 = A^H y_1 - y_1 \bar{\mu}_1$$

Let the Schur decomposition of A be

$$A = Q \begin{pmatrix} \lambda_1 & a^H \\ 0 & A_1 \end{pmatrix} Q^H, \; Q = [q_1, Q_1], \; Q^H Q = 1_n$$

Denote

$$\delta = \mathrm{sep}(\mu_1, A_1), \theta = \arccos|y_1^H q_1|, \quad 0 \le \theta \le \pi/2$$
$$\begin{pmatrix} P \\ S \end{pmatrix} = Q^H y_1, \; \Delta = \|A_1 - \mu_1 I_{n-1}\|_2, \quad D = \|A_1 - \lambda_1 I_{n-1}\|_2$$

Theorem 1.

$$(1) \sin\theta \le \|r\|_2/\delta, \quad (\text{if } \delta > 0) \tag{2.1}$$
$$(2) \|r\|_2 \le \sqrt{D^2 + \|a\|_2^2} \sin\theta. \tag{2.2}$$