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COMBINED LEGENDRE SPECTRAL-FINITE ELEMENT METHOD FOR THE TWO-DIMENSIONAL UNSTEADY NAVIER-STOKES EQUATION*

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Abstract

A combined Legendre spectral-finite element approximation is proposed for solving two-dimensional unsteady Navier-Stokes equation. The artificial compressibility is used. The generalized stability and convergence are proved strictly. Some numerical results show the advantages of this method.

Key words: Navier-Stokes equation, Combined Legendre spectral-finite element approximation.

1. Introduction

There is much literature concerning numerical solutions of Navier-Stokes equations, e.g., see [1-4]. For semi-periodic problems, some author used combined Fourier spectral-finite difference and Fourier spectral-finite element approximations (see[5-8]). In fluid dynamics, most of practical problems are fully non-periodic. But the sections of domain might be rectangular in certain directions. For example, the fluid flow in a cylindrical container. In this paper, we consider combined Legendre spectral-finite element approximation for the two-dimensional, non-periodic, unsteady Navier-Stokes equation. The method in this paper can raise the accuracy by Legendre spectral approximation in some directions and so saves work. On the other hand, such approximation is suitable for complex geometry in the remaining directions. Surely it is not necessary to use this approach for such two-dimensional problem. But it is easy to generalize it to three-dimensional problems with complex geometry.

2. The Scheme

Let $I_x = \{x\,/\,0 < x < 1\}$, $I_y = \{y\,/ -1 < y < 1\}$ and $\Omega = I_x \times I_y$ with the boundary $\partial\Omega$. The speed vector and the pressure are denoted by $U(x,y,t)$ and $P(x,y,t)$ respectively. $\nu > 0$ is the kinetic viscosity. $U_0(x,y)$, $P_0(x,y)$ and $f(x,y,t)$ are given functions. Let $T > 0, \partial_t = \frac{\partial}{\partial t}, \partial_x = \frac{\partial}{\partial x}$, and $\partial_y = \frac{\partial}{\partial y}$. The Navier-Stokes equation is as follows

$$\begin{cases} \partial_t U + (U\cdot\nabla)U + \nabla P - \nu\nabla^2 U = f, & \text{in } \Omega\times(0,T], \\ \nabla\cdot U = 0, & \text{in } \Omega\times(0,T], \\ U(x,y,0) = U_0(x,y), \quad P(x,y,0) = P_0(x,y), & \text{in } \Omega \end{cases} \tag{2.1}$$

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Suppose that the boundary is a non-slip wall and so $U = 0$ on $\partial\Omega$. In addition, P satisfies the following normalizing condition:

$$\int_\Omega P(x,y,t)\,\mathrm{d}x\mathrm{d}y = 0, \qquad \forall t \in [0,T].$$

Let $\mathcal{D}$ be an interval (or a domain) in R^1(or R^2). We denote by $(\cdot,\cdot)_{\mathcal{D}}$ and $\|\cdot\|_{\mathcal{D}}$ the usual inner product and norm of $L^2(\mathcal{D})$. For simplicity, $(\cdot,\cdot)_\Omega$ and $\|\cdot\|_\Omega$ are replaced by $(\cdot,\cdot)$ and $\|\cdot\|$ respectively. $H^r(\mathcal{D})$ and $H_0^r(\mathcal{D})$ denote the usual Hilbert spaces with the usual inner products and norms. We also define

$$L_0^2(\mathcal{D}) = \{\eta \in L^2(\mathcal{D}) \ / \ \int_{\mathcal{D}} \eta\,\mathrm{d}\mathcal{D} = 0 \ \}.$$

To tackle the incompressible constraint (i.e., the second equation of (2.1)), we adopt the idea of artificial compression, that is, to approximate the incompressible condition by the equation

$$\beta\frac{\partial P}{\partial t} + \nabla\cdot U = 0$$

where $\beta > 0$ is a small parameter.

In order to approximate the nonlinear term, we introduce a trilinear form $J(\cdot,\cdot,\cdot) : [(H^1(\Omega))^2]^3 \to R^1$ as follows:

$$J(\eta,\varphi,\xi) = \frac{1}{2}[((\varphi\cdot\nabla)\eta,\xi) - ((\varphi\cdot\nabla)\xi,\eta)].$$

Clearly, we have

$$J(\eta,\varphi,\xi) + J(\xi,\varphi,\eta) = 0, \tag{2.2}$$

and if $\nabla\cdot\varphi = 0$, then

$$J(\eta,\varphi,\xi) = ((\varphi\cdot\nabla)\eta,\xi).$$

Now we construct the scheme. For any integer $k \ge 0$, we denote by $\mathcal{P}_k$ the set of all polynomials of degree $\le k$, defined on R^1. Suppose N is a positive integer, we define

$$V_N(I_y) = \{v(y) \in \mathcal{P}_N \ / \quad v(-1) = v(1) = 0\}.$$

Next, we divide I_x into M_h subintervals with the nodes $0 = x_0 < x_1 < \cdots < x_{M_h} = 1$. Let $I_l = (x_{l-1}, x_l), h_l = x_l - x_{l-1}$, $h = \max\limits_{1\le l\le M_h} h_l$ and $h' = \min\limits_{1\le l\le M_h} h_l$. Assume that there exists a positive constant d independent of the divisions of I_x, such that $h/h' \le d$. Let

$$\tilde{S}_h^k(I_x) = \{v(x)\,/\,v(x)\,|_{I_l}\in \mathcal{P}_k, 1\le l\le M_h\}, \quad S_h^k(I_x) = \tilde{S}_h^k(I_x)\bigcap H_0^1(I_x).$$

The trial function spaces for the speed and the the pressure are defined respectively as follows

$$X_{h,N}^k(\Omega) = \{S_h^{k+1}(I_x)\otimes V_N(I_y)\} \times \{S_h^{k+2}(I_x)\otimes V_N(I_y)\},$$

$$Y_{h,N}^k(\Omega) = \{\tilde{S}_h^k(I_x)\otimes \mathcal{P}_{N-2}(I_y)\}\bigcap L_0^2(\Omega).$$