Journal of Computational Mathematics, Vol.17, No.5, 1999, 475–494.

INTERIOR ERROR ESTIMATES FOR NONCONFORMING FINITE ELEMENT METHODS OF THE STOKES EQUATIONS*

Xiao-bo Liu
(*Department of Mathematics and Computer Science Clarkson University Box 5815, Potsdam, NY 13699-5815, USA*)

Abstract

Interior error estimates are derived for nonconforming stable mixed finite element discretizations of the stationary Stokes equations. As an application, interior convergences of difference quotients of the finite element solution are obtained for the derivatives of the exact solution when the mesh satisfies some translation invariant condition. For the linear element, it is proved that the average of the gradients of the finite element solution at the midpoint of two interior adjacent triangles approximates the gradient of the exact solution quadratically.

Key words: Interior error estimates, Nonconforming element, Stokes equations.

1. Introduction

Interior error estimates for finite element discretizations (conforming) were first introduced by Nitsche and Schatz[14] for second order scalar elliptic equations in 1974. They proved that the local accuracy of the finite element approximation is bounded in terms of two factors: the local approximability of the exact solution by the finite element space and the global approximability measured in an arbitrarily weak Sobolev norm on a slightly larger domain. Since then, interior estimates of this nature have been obtained by Douglas, Jr. and Milner for mixed methods of the second order scalar elliptic equations[8], Douglas, Jr., Gupta, and Li for the hybrid method[7], by Gastaldi for a family of elements for the Reissner-Mindlin plate model[12], by Arnold and Liu for conforming finite element methods for the Stokes equations[1], and by Liu for nonconforming methods for the second order elliptic equations[13]. For a comprehensive review on this subject, see [17].

Recently, some quite interesting applications of interior estimates have been found in the areas of a posteriori error analysis and adaptive mesh refinement. In 1988 Eriksson and Johnson[11] introduced two a posteriori error estimators based on local difference quotients of the numerical solution. Their analysis was based on the interior convergence theory in [14] and [15]. In 1991, Babuška and Rodríguez[2] studied the estimators of Zhu and Zienkiewicz[19], [20] by using the interior estimate results of Bramble and Schatz[15]. For other applications in this direction, please refer to [9], [10]

* Received November 6, 1995.

and [3]. Through these investigations, it is now widely believed that the asymptotic exactness of a posteriori estimators essentially depends on some kind of superapproximation property of the finite element method. Interior error estimates, however, offer a standard approach to derive interior superconvergences.

The aim of this paper is to establish interior error estimates for nonconforming finite element approximations to solutions of the Stokes equations. Note that nonconforming methods are attractive for the Stokes problems for two reasons: (1) the inf-sup condition is easy to satisfy; (2) divergence-free nodal bases can be constructed. In addition, since the pressure can be eliminated first (when discontinuous functions are used to approximate the pressure), the velocity can be found through solving a positive system and thereafter, some preconditioned multigrid methods may be incorporated for constructing fast solvers.

The method used here and the structure of this paper closely follows that in [1]. Section 2 presents notations and preliminaries. Section 3 introduces hypotheses for the finite element spaces, which actually apply for both nonconforming and conforming methods. In Section 4, we introduce the interior equations and derive some basic properties of their solutions. Section 5 gives the precise statement of our main result and its proof. In Section 6 we prove interior convergences of difference quotients of the finite element solution to the derivatives of the exact solution when the finite element space is defined over meshes with certain translation invariant property. An interior superconvergence is obtained as an example application.

2. Notations and Preliminaries

Let Ω denote a bounded domain in $\mathbb{R}^2$ and $\partial\Omega$ its boundary. We shall use the usual standard L^2-based Sobolev spaces $H^m = H^m(\Omega)$, $m \in \mathbb{Z}$, with the norm $\|\cdot\|_{m,\Omega}$. Recall that for $m \in \mathbb{N}$, H^{-m} denotes the normed dual of $\overset{\circ}{H}{}^m$, the closure of $C_0^\infty(\Omega)$ in H^m. We use the notation $(\cdot,\cdot)$ for both the $L^2(\Omega)$-innerproduct and its extension to a pairing of $\overset{\circ}{H}{}^m$ and H^{-m}. If $\overline{\Omega} = \bigcup\limits_j \overline{\Omega}_j$ for some disjoint open sets Ω_j, then let $H_h^m(\Omega) = \{u \in L^2(\Omega)$ and $u|_{\Omega_j} \in H^m(\Omega_j)$, for all $j\}$ with the norm $\|u\|_{m,\Omega}^h = \Big(\sum\limits_j \|u\|_{m,\Omega_j}^2\Big)^{1/2}$. If X is any subspace of L^2, then $\hat{X}$ denotes the subspace of elements with average value zero. We use boldface type to denote 2-vector-valued functions, operators whose values are ector-valued or tensor-valued functions, and spaces of vector-valued functions. This is illustrated in the definitions of the following standard differential operators:

$$\operatorname{div}\phi = \partial\phi_1/\partial x + \partial\phi_2/\partial y,\ \textbf{grad}\ p = \begin{pmatrix} \partial p/\partial x \\ \partial p/\partial y \end{pmatrix},\ \textbf{grad}\ \phi = \begin{pmatrix} \partial\phi_1/\partial x & \partial\phi_1/\partial y \\ \partial\phi_2/\partial x & \partial\phi_2/\partial y \end{pmatrix}.$$

For any function ϕ that is differentiable on each Ω_i where $\overline{\Omega} = \bigcup\limits_i \overline{\Omega}_i$, a family of disjoint open sets Ω_i, we define the piecewise version (with notation div_h) of its divergence to be the function obtained by computing $\operatorname{div}\phi$ element-wise. The piecewise version of the gradient operator can be defined similarly and is denoted by grad_h.