Journal of Computational Mathematics, Vol.17, No.4, 1999, 369-378.

A ESTIMATE OF THE RATE OF ENTROPY DISSIPATION OF HIGH RESOLUTION MUSCL TYPE GODUNOV SCHEMES*1)

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Abstract

In this paper, following the paper [7], we analysis the "sharp" estimate of the rate of entropy dissipation of the fully discrete MUSCL type Godunov schemes by the general compact theory introduced by Coquel–LeFloch [1, 2], and find: because of small viscosity of the above schemes, in the vincity of shock wave, the estimate of the above schemes is more easily obtained, but for rarefaction wave, we must impose a "sharp" condition on limiter function in order to keep its entropy dissipation and its convergence.

Key words: Hyperbolic conservation laws, MUSCL type godunov schemes, Entropy dissipation, shock wave, Rarefaction wave.

1. Introduction

Let us consider the Cauchy problems for nonlinear hyperbolic scalar conservation laws:

$$\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} = 0, \tag{1.1}$$

$$u(x,0) = u_0(x) \tag{1.2}$$

where $f : \Re \longrightarrow \Re$ is Lipschitz continuous functions, and the initial data $u_0(x)$ is a given function in $L^1(R) \bigcap L^\infty(R)$. As it is well-known, this problem in general does not admit smooth solution, so that weak solutions in the sense of distributions must be consider.

* Received December 6, 1995.

1) This work was supported by the China National Aeronautical Foundation and by the China Postdoctoral Science Foundation.

Moreover, an entropy condition must be added in order to ensure the uniqueness of the weak solutions of equation (1.1) and (1.2). The convergence of high resolution schemes has been investigated by many authers, such as Osher and Tadmor [3], Vila [5],and Coquel and LeFloch [1, 2]. However, some quantities depending on space mesh size are always introduced in their paper. In general, the difference schemes only depends on the ratio of the mesh size but the mesh size. So, the introduction of these quantities may be improper.

In this paper, we discuss a class of the fully discrete MUSCL type Godunov schemes based on the general theory introduced by Coquel-LeFloch [1, 2]. In section 2, we recall the Godunov schemes for scalar conservation laws and give its MUSCL type high resolution Godunov schemes. Section 3 deals with the rate of entropy dissipation of the schemes. We give a cubic estimate of the "sharp" entropy inequalities of the MUSCL type Godunov schemes in the case of shock wave. Moreover, we anaylsis the case of rarefaction wave, and find that a "sharp" condition must be imposed on the limiter functions in the case of rarefaction wave, in order to ensure entropy inequalities and convergence of the above schemes. The above limitations make these schemes fail to preserve the second order accuracy.

2. The Fully Discrete MUSCL Type Godunov Schemes

Let us consider finite difference schemes in conservative form for conservation laws (1) and (2)

$$u_j^{n+1} = u_j^n - \lambda(h_{j+\frac{1}{2}} - h_{j-\frac{1}{2}}), \tag{2.1}$$

where $\lambda = \Delta t/\Delta x$ is the mesh ratio, and Δt and Δx are the variable meshsize in time and space directions, respectively. $h_{j+\frac{1}{2}}$ denotes the numerical flux

$$h_{j+\frac{1}{2}} = h(u_{j-s+1}, ..., u_{j+s}), h(u, ..., u) = f(u). \tag{2.2}$$

As well known, the weak solution of equation (1.1) and (1.2) is not unique. So let the function U(u) be any convex function,so–called the entropy function, and associated with entropy function F(u) satisfies $F'(u) = U'(u)f'(u)$. (U,F) is called an entropy pair. If the weak solution of equation (1.1) and (1.2) satisfies the following inequality:

$$\frac{\partial U(u)}{\partial t} + \frac{\partial F(u)}{\partial x} \leq 0, \tag{2.3}$$

in the sense of distribution to every entropy pair (U,F), the weak solution is the unque physical solution of equation (1.1) and (1.2). The inequality (2.3) is called the entropy inequality (or the entropy condition). Corresponding to the conservative scheme (2.1), the discrete entropy inequality is defined as

$$U(u_j^{n+1}) - U(u_j^n) + \lambda(H_{j+\frac{1}{2}} - H_{j-\frac{1}{2}}) \leq 0, \tag{2.4}$$