Journal of Computational Mathematics, Vol.17, No.2, 1999, 133–138.

A LEAP FROG FINITE DIFFERENCE SCHEME FOR A CLASS OF NONLINEAR SCHRÖDINGER EQUATIONS OF HIGH ORDER[*1)]

Wen-ping Zeng
(*Department of Mathematics, Overseas Chinese University, Quanzhou 362011, China*)

Abstract

In this paper, the periodic initial value problem for the following class of nonlinear Schrödinger equation of high order

$$i\frac{\partial u}{\partial t} + (-1)^m \frac{\partial^m}{\partial x^m}\Big(a(x)\frac{\partial^m u}{\partial x^m}\Big) + \beta(x)q(|u|^2)u + f(x,t)u = g(x,t)$$

is considered. A leap-frog finite difference scheme is given, and convergence and stability is proved. Finally, it is shown by a numerical example that numerical result is coincident with theoretical result.

Key words: High order nonlinear Schrödinger equation, Leap-Frog difference scheme, Convergence.

1. Introduction

It is well know that the nonlinear equations of Schrödinger type are of great importance to physics and can be used to describe extensive physical phenomena[1].

In this paper,we will consider the periodic initial value problem for the following class of nonlinear Schrödinger equation of high order:

$$\begin{cases} i\dfrac{\partial u}{\partial t} + (-1)^m \dfrac{\partial^m}{\partial x^m}\Big(a(x)\dfrac{\partial^m u}{\partial x^m}\Big) + \beta(x)q(|u|^2)u + f(x,t)u = g(x,t) & (x,t) \in R\times I \qquad (1.1) \\ u|_{t=0} = u_0(x) & x \in R \qquad (1.2) \\ u(x+L,t) = u(x,t) & (x,t)\in R\times I \qquad (1.3) \end{cases}$$

where $i = \sqrt{-1}$, $R = (-\infty, +\infty)$, $I = [0, T]$, $u \equiv u(x,t)$ is an unknow complex valued function of x with period L, and $\overline{u}$ is a conjugate complex function of u; $f(x,t), g(x,t)$, $a(x)$ and $\beta(x)$ are all real-valued function x with period L; $u_0(x)$ is given complex-valued function with period L; $q(\cdot)$ is a continuous real-valued function with real variable, and compound function $z \to q^*(z) = q(|z|^2)$ exist a continuous partial derivative to Rez, Imz. Besides, suppose the following conditions are true:

$$\begin{cases} 0 < m' \le a(x) \le M \\ \max\limits_{(x,t)\in R\times I}\{|\beta(x)|, |f(x,t)|\} = M_1 \end{cases} \qquad (A)$$

* Received August 2, 1994.
1) Supported by Fujian Natural Science Foundation.

where m',M and M_1 are all positive constant.

In the paper [2], there have discussed initial Value problem of system such as (1.1)–(1.3), introduced a difference scheme of conservation type, and researched its stability and convergence. Otherwise, it is an implicit method and its difference scheme is a nonlinear system.

In this paper, we introduce a leap-frog finite difference scheme for the periodic initial value problem (1.1)–(1.3) of a class of nonlinear Schrödinger equation of high order, its difference scheme is explicit, easily solved. Its convergence and stability can be proved.

Finally, it is shown by a numerical example that numerical result is coincident with theoretical result.

2. Establishment of the Difference Scheme

First we introduce some notations. Let $Q_T = [0.L] \times I$ be a rectangular region, where $I = [0,T]$. We divide the domain Q_T into small grids by the parallel lines $x = x_j = jh$, $t = t_n = nk$ $(j = 0,1,\cdots,J;\ n = 0,1,\cdots,N)$, where $Jh = L$, $N = \left[\dfrac{T}{k}\right]$. Let $Q_h = \{(x,t);\ x = jh,\ t = nk,\ j = 0,1,\cdots,J;\ n = 0,1,\cdots,N)$. And Let ϕ_j^n $(j = 0,1,\cdots,J;\ n = 0,1,\cdots,N)$ denote the discrete function on the grid point (x_j,t_n).

Define

$$\Delta_+\phi_j^n = \phi_{j+1}^n - \phi_j^n, \qquad \Delta_-\phi_j^n = \phi_j^n - \phi_{j-1}^n$$
$$D_t\phi_j^n = \frac{1}{2k}(\phi_j^{n+1} - \phi_j^{n-1}), \qquad \delta^{2m}\phi_j^n = \Delta_+^m(a_{j-\frac{m}{2}}\Delta_-^m\phi_j^n)h^{-2m}$$

where $a_{j-\frac{m}{2}} = a((j-\frac{m}{2})h)$, ϕ_j^ndenote the discrete function value on the grid point (jh,nk).

We also introduce the inner product and norms appropriate to function defined on the lattice Q_h, i.e

$$(v,w) = (v,w)_h = h\sum_{j=1}^{J} v(x_j)\overline{w}(x_j) \quad \forall v,w \in c^J$$
$$||v||^2 = ||v||_h^2 = (v,v)_h = (v,v)$$

where C^J is a J-dimensionally complex space.

Corresponding to (1.1)–(1.3),we construct following leap-frog finite difference scheme

$$\begin{cases} iD_t\phi_j^n + (-1)^m\delta^{2m}\phi_j^n + \beta_j q(|\phi_j^n|^2)\phi_j^n + f_j^n\phi_j^n = g_j^n \\ \qquad\qquad (j = 1,2,\cdots,J; n = 1,2,\cdots,N = ([T/k]) & (2.1) \\ \phi_j^0 = U_0(jh) \qquad (j = 1,2,\cdots,J) & (2.2) \\ \phi_{rJ+j}^n = \phi_j^n \begin{cases} j = 1,2,\cdots,J; r = \pm 1, \pm 2,\cdots \\ n = 0,1,\cdots,N \end{cases} & (2.3) \end{cases}$$

In difference scheme (2), if ϕ_j^1 $(j = 1,2,\cdots,J)$ is given, it can be calculated level by level explicitly. And ϕ_j^1 can calculate by the scheme with same convergence order of the scheme (2), example conservation type difference scheme in [2].