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CONVERGENCE OF CHORIN-MARSDEN FORMULA FOR THE NAVIER-STOKES EQUATIONS ON CONVEX DOMAINS*

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Abstract

We prove the convergence of the Chorin-Marsden product formula for solving the initial-boundary value problems of the Navier-Stokes equations on convex domains. As a particular case we consider the case of the half plane.

Key words: Navier Stokes equation, Vortex method, Fractional step method, Convergence

1. Introduction

Different kinds of fractional step methods have been applied to solve the initial-boundary value problems of the Navier-Stokes equations for viscous incompressible flow. The vortex method developed in [5] by Chorin is a scheme with three intermediate steps where the effects of convection and viscosity are separated, and vortex sheets are created along the boundary. A set of vortex blobs is introduced to approximate the vorticity field. These vortex blobs move along the particle trajectories in the convection step, and they move randomly in the diffusion step. The convergence of the scheme is an interesting problem which has called the attention of many authors.

Related to this scheme, the splitting of the initial-boundary value problems of the Navier-Stokes equations to the corresponding problems of the Euler equations and the Stokes equations has been extensively studied, see [2] [3] [7] [9] [10] [11] [12] [13] [14] and the references therein. By the results a simple splitting converges in $L^p, p < \infty$, and in $H^s, s < \frac{5}{2}$, and if the vortex sheets are smeared out such that the vorticity is smooth, then the scheme with some modification still converges.

Marsden gave one mathematical formulation of Chorin's scheme which is a product of three operators,

$$u_k(ik) = (H_k \circ \phi \circ E_k)^i u_0,$$

where u_0 is the initial data, E_k is the local flow defined by the Euler equations with temporal step k, ϕ is the "vorticity creation operator", and H_k is the solver of the heat equation with step k. This formula is known as the Chorin-Marsden formula[6]. It involves a further approximation beyond the splitting. In [6] the velocity field is extended oddly to the exterior of the domain and the Cauchy problem of the heat

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equation for the velocity is solved in the diffusion step rather than the initial-boundary value problem of the Stokes equation . This approximation is consistent to the random walk procedure. Convergence of the linear problems was proved in [6]. Benfatto and Pulvirenti studied the Chorin-Marsden formula in the case of the half plane for the Navier-Stokes equations and proved the convergence[4]. The scheme in [4] is different from that in [6] by two respects: The tangential component of the velocity is also extended oddly but the normal component is extended evenly, and an explicit Euler scheme is applied in the convection step rather than using the particle method. The first modification bears the advantage that the velocity field keeps incompressible after the extension.

The purpose of this paper is to prove the convergence of the Chorin-Marsden formula for arbitrary two dimensional convex domains. In the convection step we use the velocity of the previous step to slove the particle trajectories, making the step in fact linear. In the diffusion step we use a modified approach to extend the velocity. Particularly if the domain is the half plane then the extension here is the same as that in [6].

In section 2 we state the scheme in details and introduce some notations. In section 3 we prove the convergence of the scheme for convex domains, where for simplicity we assume that the domains are bounded. In section 4 we apply our approach to the case of the half plane, and we will show that both approaches of extension, by Chorin-Marsden and by Benfatto and Pulvirenti, yield the results of convergence.

2. Scheme

Let $\Omega \subset R^2$ be a domain with sufficiently smooth boundary $\partial\Omega$ and $x = (x_1, x_2)$ be the points in R^2. We consider the following initial-boundary value problems,

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \frac{1}{\rho}\nabla p = \nu \triangle u + f, \tag{1}$$

$$\nabla \cdot u = 0, \tag{2}$$

$$u|_{\partial\Omega} = 0, \tag{3}$$

$$u|_{t=0} = u_0, \tag{4}$$

where $u = (u_1, u_2)$ is the velocity, p is the pressure, f is the external force, ρ is the constant density, ν is the constant kinematic viscosity, and $\nabla = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})$. We introduce the vorticity $\omega = -\nabla \wedge u$ and the stream function ψ such that $u = \nabla \wedge \psi$, where $\nabla\wedge = (\frac{\partial}{\partial x_2}, -\frac{\partial}{\partial x_1})$, then the vorticity-stream function formulation of the problems is

$$\frac{\partial \omega}{\partial t} + u \cdot \nabla\omega = \nu \triangle \omega + F, \tag{5}$$

$$-\triangle\psi = \omega, \quad \psi|_{\partial\Omega} = 0, \quad \left.\frac{\partial\psi}{\partial n}\right|_{\partial\Omega} = 0, \tag{6}$$

$$u = \nabla \wedge \psi, \tag{7}$$

$$\omega|_{t=0} = \omega_0, \tag{8}$$