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NEW APPROACH TO THE LIMITER FUNCTIONS*

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Abstract

In this paper we discuss three topics on the designing of the limiter functions. (1) To guarantee the TVD property (2) To maintain enough artificial viscosity. (3) A method to form TVB limiter which can ensure second order accuracy even at the extrema of the solution.

Key words: Finite difference methods, TVD schemes, Limiter Function

1. Introduction

Since 1980's, difference schemes with TVD or TVB properties have been used for more and more CFD problems, especially the following system of conservation laws:

$$U_t + F(U)_x = 0. \tag{1.1}$$

The reason is that the TVB property will guarantee the convergence of any subsequence of the difference solution sequence to a week solution of the differential equation. Obviously if the week solution is unique, then the whole sequence will converge to that solution.

One of the frequently used TVD scheme is the second order five-point conservative one:

$$U_i^{n+1} = U_i^n - \lambda(H_{i+1/2} - H_{i-1/2}). \tag{1.2}$$

Here $H_{i+1/2} = H(U_{i-1}^n,\ U_i^n,\ U_{i+1}^n,\ U_{i+2}^n)$, is consistent with F, i.e, $H(U,U,U,U) = F(U)$, and could be written as

$$H_{i+1/2} = F(U_i^n) + Q_{i+1/2} \cdot (F(U_{i+1}^n) - F(U_i^n)), \tag{1.3}$$

here $Q_{i+1/2}$ is usually a nonlinear function of $U_{i-1}^n, \cdots, U_{i+2}^n$, and is called Limiter.

It is this Limiter that has great effect on the scheme. In this paper we will discuss some principles and methods on how to construct that function in order that the scheme has desired properties. For simplicity, we begin with the following scalar linear equation as the model problem:

$$U_t + a \cdot U_x = 0. \tag{1.4}$$

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The corresponding scheme is:

$$U_i^{n+1} = U_i^n - a \cdot \lambda(U_i^n + Q_{i+1/2} \cdot \Delta U_{i+1/2} - U_{i-1}^n - Q_{i-1/2} \cdot \Delta U_{i-1/2}) \tag{1.5}$$

here $\lambda = \frac{\Delta t}{\Delta x}$, $\Delta U_{i+1/2} = U_{i+1}^n - U_i^n$. Without loss of generality, we assume here $a \geq 0$.

Although the above simple model is used for the theoretical analysis, the background problem of this paper is a practical 3-D viscous outer flow one, so some of the numerical examples are about 3-D flow problems.

In section 2, the conditions on Limiter for TVD property are discussed. In section 3, for solving the problems arising in the practical flow calculation, some ideas on maintaining proper artificial viscosity are given. In section 4, a method for constructing a Limiter which will ensure the second order accuracy of the scheme even at the extrema of the solution while keeping the TVB property is presented. The results of numerical experiments are provided in section 5.

2. The Basic Conditions for TVD Limiters

Accordiong to the TVD sufficient condition of Harten in [1], if a scheme can be written as:

$$U_i^{n+1} = U_i^n + C_{i+1/2}^+ \Delta U_{i+1/2} - C_{i-1/2}^- \Delta U_{i-1/2} \tag{2.1}$$

and if

$$C_{i+1/2}^+, C_{i+1/2}^- \geq 0, \quad C_{i+1/2}^+ + C_{i+1/2}^- \leq 1 \tag{2.2}$$

then the scheme is a TVD one.

The scheme (1.5) can be put into the form (2.1) if we choose:

$$C_{i-1/2}^- = a \cdot \lambda(1 + Q_{i+1/2} \frac{\Delta U_{1+1/2}}{\Delta U_{i-1/2}} - Q_{i-1/2}), \quad C_{i+1/2}^+ = 0. \tag{2.3}$$

Assume that $Q_{i+1/2}$ is a function of the difference ratio $r_{i+1/2} = \frac{\Delta U_{i-1/2}}{\Delta U_{i+1/2}}$, i.e, $Q_{i+1/2} = Q(r_{i+1/2})$, and the function satisfies:

$$\begin{array}{lcl} Q(r) = 0 & & r \leq 0 \\ 1 \geq Q(r) > 0 & \text{if} & r > 0\,. \\ Q(r) = 1/2 & & r = 1 \end{array} \tag{2.4}$$

Furthermore we require the $Q(r)$ is Lipschitz continuous, i.e, there is a $L > 0$ independent of r, such that for any r, r':

$$|Q(r) - Q(r')| \leq L \cdot |r - r'|. \tag{2.5}$$

Thus, there must be $Q(0) = 0$, for any $r > 0$:

$$|Q(r)| = |Q(r) - Q(0)| \leq L \cdot r. \tag{2.6}$$

Therefore, when $a \cdot \lambda \leq \frac{1}{1+L}$, for the coefficient $C_{i-1/2}^-$ in (2.3), we have:

$$\text{if} \quad \frac{\Delta U_{i+1/2}}{\Delta U_{i-1/2}} \leq 0: \tag{2.7}$$