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NONCONFORMING FINITE ELEMENT APPROXIMATIONS TO THE UNILATERAL PROBLEM[*1)]

Lie-heng Wang
(*State Key Laboratory of Scientific and Engeneering Computing, ICMSEC, Chinese Academy of Sciences, Beijing 100080, China*)

Abstract

The nonconforming finite element (two Crouzeix-Raviart linear elements and Wilson element) approximations to the unilateral problem are considered. The error bounds for these elements are obtained in the appropriate assumptions of regularity of solution of the problem.

Key words: Unilateral problem, Nonconforming finite element

1. Introduction

There have been numerous work in the analysis of finite element methods for the unilateral problem (c.f.[4] and the references therein). It should be mentioned that in F. Scarpini et. al.[6], I.Hlavacek et. al.[5] and F. Brezzi et. al.[1], the conforming linear element approximation to the unilateral problem have been considered, and the various error bounds have been obtained in the different assumption of regularity of solution of the problem.

In this paper, we consider three nonconforming finite element (i.e. two Crouzrix-Raviart linear elements and Wilson element) approximations to the unilateral problem, and the error bounds for these elements are obtained in the appropriate assumptions of regularity of solution of the problem.

The unilateral problem is the following

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_0, \\ u \geq 0,\ \partial u/\partial \nu \geq 0,\ u\partial u/\partial \nu = 0, & \text{on } \Gamma_1, \end{cases} \tag{1.1}$$

where Ω is a convex domain in R^2 with piecewise smooth boundary $\partial\Omega$, $\partial\Omega = \Gamma_0 \cup \Gamma_1$, $\Gamma_0 \cap \Gamma_1 = \emptyset$ and $\partial u/\partial \nu$ is the outer normal derivative of u on Γ_1. It is well known that the problem (1.1) is equivalent to the following variational inequality:

$$\begin{cases} \text{to find } u \in K, & \text{such that} \\ a(u, v-u) \geq \ \langle f, v-u \rangle & \forall v \in K, \end{cases} \tag{1.2}$$

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where

$$K = \{v \in H^1(\Omega) |\ v = 0 \text{ on } \Gamma_0, v \geq 0 \text{ on } \ \Gamma_1\}, \tag{1.3}$$

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v dx, \quad \langle f, v\rangle = \int_\Omega f \cdot v dx. \tag{1.4}$$

the solution of the problem (1.2) will be approximated by the finite element method with a regular subdivision. For each $h > 0$, let $\mathcal{T}_h$ be a regular subdivision of Ω. For the sake of simplisity, let Ω be a convex polygon, then $\Omega = \bigcup_{\tau \in \mathcal{T}_h} \tau$. Let V_h be a finite element space of approximating the space $H^1_{\Gamma_0}(\Omega) = \{v \in H^1(\Omega) | v = 0 \text{ on } \Gamma_0\}$, with norm $\|\cdot\|$:

$$\|v\|_h = \Big(\sum_{\tau \in \mathcal{T}_h} |v|^2_{1,\tau}\Big)^{\frac{1}{2}} \quad \forall\ v \in V_h, \tag{1.5}$$

and K_h be a convex closed subset of V_h, as an approximation of K. Then the approximate problem of the unilateral problem (1.2) is the following:

$$\begin{cases} \text{to find} \quad u_h \in K_h, & \text{such that} \\ a_h(u_h, v_h - u_h) \geq \ \langle f, v_h - u_h\rangle & v_h \in K_h, \end{cases} \tag{1.6}$$

where

$$a_h(u_h, v_h) = \sum_{\tau \in \mathcal{T}_h} \int_\tau \nabla u_h \nabla v_h dx. \tag{1.7}$$

We now show abstract error estimate

Theorem 1.1. *Assume that u and u_h are the solutions of the problems* (1.2) *and* (1.6) *respectively, then*

$$\|u - u_h\|_h^2 \leq C \inf_{v_h \in K_h} \{\|u - v_h\|^2_h + a_h(u, v_h - u_h) - \langle f, v_h - u_h\rangle\}. \tag{1.8}$$

Proof. Using the triangle inequality

$$\|u - u_h\|_h \leq \|u - v_h\|_h + \|v_h - u_h\|_h, \quad \forall\ v_h \in K_h.$$

And noting that u_h is the solution of the problem (1.6),

$$\begin{aligned} \|v_h - u_h\|^2_h &= a_h(v_h - u_h, v_h - u_h) \\ &= a_h(v_h - u, v_h - u_h) + a_h(u - u_h, v_h - u_h) \\ &\leq \|v_h - u\|_h \cdot \|v_h - u_h\|_h + a_h(u, v_h - u_h) - \langle f, v_h - u_h\rangle. \end{aligned}$$

Summarizing the previous two inequalities, the theorem is proved.

2. Crouzeix-Raviart Linear Element Approximation(I)

For the Crouzeix-Raviart linear element approximation to the unilateral problem (1.2), the subdivision $\mathcal{T}_h$ is a triangulation, $\tau \in \mathcal{T}_h$ triangle element,

$V_h = \{v_h \in L^2(\Omega) : v_h|_\tau \in P_1(\tau),\ v_h$ is continuous at the midpoints of edges of element