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ON NUMEROV SCHEME FOR NONLINEAR TWO-POINTS BOUNDARY VALUE PROBLEM*

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Abstract

Nonlinear Jacobi iteration and nonlinear Gauss-Seidel iteration are proposed to solve the famous Numerov finite difference scheme for nonlinear two-points boundary value problem. The concept of supersolutions and subsolutions for nonlinear algebraic systems are introduced. By taking such solutions as initial values, the above two iterations provide monotone sequences, which tend to the solutions of Numerov scheme at geometric convergence rates. The global existence and uniqueness of solution of Numerov scheme are discussed also. The numerical results show the advantages of these two iterations.

Key words: Nonlinear two-points boundary value problem, New iterations for Nomerov scheme, Monotone approximations.

1. Introduction

In studying some problems arising in electromagnetism, biology, astronomy, boundary layer and other topics, we often meet nonlinear two-points boundary problem, i.e., finding $y \in C^0[0,1] \cap C^2(0,1)$ such that

$$\begin{cases} -y'' - f(x, y(x)) = 0, \quad 0 < x < 1, \\ y(0) = \alpha, \ y(1) = \beta \end{cases} \tag{1.1}$$

where α, β are certain constants, and $f(x,z) \in C^0(0,1) \times C^1(-\infty,\infty)$. Under some conditions on $f(x,z)$, we can use the framework of [1] to investigate the existence and uniqueness of its solutions. Also there are a lot of literature concerning its numerical solutions[2–4]. In particular, Numerov[5] proposed a famous finite difference scheme with the accuracy of fourth order, which has been used widely in many practical problems. Let N be any positive integer and $h = \dfrac{1}{N}$, $x_n = nh$, $0 \le n \le N$. Also, let $y_n = y(x_n)$, $f_n = f(x_n, y_n)$, and

$$\begin{aligned} &Y = (y_1, \cdots, y_{N-1})^T, \quad F(Y) = (f_1, \cdots, f_{N-1})^T, \\ &C = (\alpha, 0, \cdots, 0, \beta)^T, \quad D = \Big(\frac{1}{12} f(0,\alpha), 0, \cdots, 0, \frac{1}{12} f(1,\beta)\Big)^T. \end{aligned}$$

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Moreover we introduce the symmetric tridiagonal matrices $J = (J_{i,j})$ and $B = (B_{i,j})$ with the following elements $J_{i,i} = 2$, $J_{i,i-1} = J_{i,i+1} = -1$, $1 \le i \le N-1$, $B_{i,i} = 5/6$, $B_{i,i-1} = B_{i,i+1} = 1/12$, $1 \le i \le N-1$. Then the Numerov scheme can be described as follows[5]

$$L_h(Y) \equiv JY - h^2(BF(Y) + D) - C = 0. \tag{1.2}$$

If $f(x, y)$ is nonlinear in y, then we need some iterations to solve (1.2). Henrici[6] and Less[7] considered the Newton iteration. Chawla[8] improved the results of [6,7]. He proposed a suitable initial approximation of the Newton procedure and obtained the sufficient conditions for the convergence when $-\infty < \dfrac{\partial f}{\partial z}(x, z) < \pi^2$. But such conditions involve an implicit equation for the mesh size h and it is difficult to solve it usually. In addition, we have to adopt an interior iteration for solving a linear system for each step of the exterior iteration, which costs a lot of computational time. The purpose of this paper is to develop two new iterations. In next section, we introduce nonlinear Jacobi iteration and nonlinear Gauss-Seidel iteration. Both of them avoid the interior iterations in [8], and so save a lot of work. Also, we introduce the concept of supersolutions and subsolutions, and prove that if we take such solutions as initial values, then the above iterations may provide two monotone sequences. They not only give us the up-bound and low-bound of the exact solution of (1.2), but also tend to it with geometric convergence rates. In Section 3, we consider global existence and uniqueness of solution of (1.2) as well as the global convergences of the new iterations. In the final section, we present the numerical results which agree the theoretical analysis and show the advantages of the two new approaches.

2. New Nonlinear Iterations

We now present nonlinear Jacobi iteration and nonlinear Gauss-Seidel iteration for (1.2). Let ω be a parameter. We decompose the matrices J and B as $J = \mathcal{D} - \mathcal{L} - \mathcal{U}$, $B = \mathcal{D}^* + \mathcal{L}^* + \mathcal{U}^*$, where $\mathcal{D}$ and $\mathcal{D}^*$ are diagonal matrices, $\mathcal{L}$ and $\mathcal{L}^*$ are lower-off diagonal matrices, $\mathcal{U}$ and $\mathcal{U}^*$ are upper-off diagonal matrices. Let $Y^{(m)}$ be the m'th iterated vector $(y_1^{(m)}, \cdots, y_{N-1}^{(m)})^T$ and $y_i^{(m)} = y^{(m)}(x_i)$. Then the nonlinear Jacobi iteration is defined as

$$(\mathcal{D} - \omega h^2 \mathcal{D}^*)Y^{(m)} = (\mathcal{L} + \mathcal{U})Y^{(m-1)} - \omega h^2 \mathcal{D}^* Y^{(m-1)} + h^2 BF(Y^{(m-1)}) + h^2 D + C, \tag{2.1}$$

while the Gauss-Seidel iteration is given by

$$(\mathcal{D} - \mathcal{L} - \omega h^2(\mathcal{D}^* + \mathcal{L}^*))Y^{(m)} = \mathcal{U}Y^{(m-1)} - \omega h^2(\mathcal{D}^* + \mathcal{L}^*)Y^{(m-1)} + h^2 BF(Y^{(m-1)}) + h^2 D + C. \tag{2.2}$$

Clearly both (2.1) and (2.2) do not need the interior iterations to solve $Y^{(m)}$ as long as $Y^{(m-1)}$ is known.

For theoretical analysis, we first introduce some notations and analyze the monotonicity of the matrix $J - \omega h^2 B$. Let $U = (u_1, \cdots, u_{N-1})^T$ and $V = (v_1, \cdots, v_{N-1})^T$. If $u_i \le v_i$ for all i, then we say that $U \le V$. If $U \le W \le V$, then it is denoted by $W \in \mathbf{K}(U, V)$. If all elements of a vector U or a matrix $A = (A_{i,j})$ are non-negative, then we say that $U \ge 0$ or $A \ge 0$, etc.. Furthermore if $AU \ge 0$ implies $U \ge 0$ for any