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SPLITTING A CONCAVE DOMAIN TO CONVEX SUBDOMAINS*

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Abstract

We examine a steady-state heat radiation problem and its finite element approximation in R^d, $d = 2, 3$. A nonlinear Stefan-Boltzmann boundary condition is considered. Another nonlinearity is due to the fact that the temperature is always greater or equal than $0[K]$. We prove two convergence theorems for piecewise linear finite element solutions.

Keywords: Nonlinear elliptic boundary value problems, heat radiation problem, finite elements, variational inequalities.

1. Introduction

It is known from physics that a body loses heat energy from its surface by electromagnetic waves. This phenomenon is called radiation[8,20]. The energetical losses are proportional to the fourth power of the surface temperature (the Kirchhoff law). Thus the radiation cannot be neglected when the surface temperature is high (e.g., in computation of temperature distribution in large dry transformers, electrical engines, $\cdots$). It is represented by the nonlinear boundary condition $\alpha(u - u_0) + n^\top \mathcal{A}$ grad $u + \beta(u^4 - u_0^4) = \tilde{g}$, where $\alpha \geq 0$ is the coefficient of convective heat transfer, u is the temperature of the body, u_0 is the surrounding temperature, n is the outward unit normal to the surface, $\mathcal{A}$ is a symmetric uniformly positive definite matrix of heat conductivities, $\beta = \sigma f_{\rm em}$, $\sigma = 5.669 \times 10^{-8}[\mathrm{Wm}^{-2}\mathrm{K}^{-4}]$ is the Stefan-Boltzmann constant, $0 \leq f_{\rm em} \leq 1$ is the relative emissivity function and $\tilde{g}$ is the density of surface heat sources.

Consider the following classical formulation of the radiation problem: Find $u \in C^2(\overline{\Omega})$, $u \geq 0$, such that

$$\begin{aligned} -\mathrm{div}\,(\mathcal{A} \text{ grad } u) &= f \quad \text{in } \Omega, \\ u &= \overline{u} \quad \text{on } \Gamma_1, \\ \alpha u + n^\top \mathcal{A} \text{ grad } u + \beta u^4 &= g \quad \text{on } \Gamma_2, \end{aligned} \tag{1.1}$$

where $\Omega \subset R^d$, $d = \{2, 3\}$, is a bounded domain with a Lipschitz-continuous boundary $\partial\Omega$, Γ_1 and Γ_2 are non-empty disjoint sets, which are relatively open in $\partial\Omega$, and satisfy

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$\partial\Omega = \overline{\Gamma}_1 \cup \overline{\Gamma}_2$, f is the density of body heat sources, $\overline{u} \geq 0$ is the prescribed temperature and $g = \tilde{g} + \alpha u_0 + \beta u_0^4$.

Similar heat radiation problems were investigated by many authors (see, e.g., [3, 8, 18, 19, 21, 22]). A proof of the existence and uniqueness of the classical solution is given in [2] for a regular boundary.

Throughout the paper we use the standard Sobolev space notation [14, 15, 17]. We will introduce a variational inequality approach to the problem (1.1) and examine its finite element approximation under the maximum angle condition. We also generalize some results of [16, 21] for $d = 2$ to the three-dimensional space.

2. Variational Formulation of a Two-Dimensional Problem

Since the classical solution of the problem (1.1) need not exist, we introduce its variational formulation. To this end we suppose that the entries of $\mathcal{A}$ belong to $L^\infty(\Omega)$, $f \in L^2(\Omega)$, $\overline{u} \in H^1(\Omega)$, $\alpha, \beta \in L^\infty(\Gamma_2)$ and $g \in L^2(\Gamma_2)$. Introduce a space of test functions $V = \{v \in H^1(\Omega) | v = 0 \text{ on } \Gamma_1\}$ and a set $U = \{v \in H^1(\Omega) | v \geq 0 \text{ in } \Omega,\ v = \overline{u}$ on $\Gamma_1\}$. It is easy to verify that U is convex, closed with respect to the norm $\|.\|_1$ and nonempty as $\overline{u} \in U$. Note that it has no interior points. (To see this for $d = 2$ and $(0,0) \in \Omega$, a simple example can be constructed using the function

$$v_\varepsilon(x_1, x_2) = -\varepsilon\big(-\ln\sqrt{x_1^2 + x_2^2}\big)^{1/4}, \quad (x_1, x_2) \in \Omega, \quad \varepsilon > 0,$$

which has a negative pole and $\|v_\varepsilon\|_1 \to 0$ for $\varepsilon \to 0$, compare [14, p. 10]).

Define a symmetric bilinear continuous form

$$a(v, w) = \int_\Omega (\text{grad } v)^\top \mathcal{A} \text{ grad } w dx + \int_{\Gamma_2} \alpha v w ds, \quad v, w \in H^1(\Omega),$$

and a linear continuous form

$$F(v) = \int_\Omega f v dx + \int_{\Gamma_2} g v ds, \quad v \in H^1(\Omega).$$

Using positive definiteness of $\mathcal{A}$, the Friedrichs inequality and the fact that $\Gamma_1 \neq \emptyset$, we obtain the V-ellipticity of $a(.,.)$,

$$a(v, v) \geq C\|v\|_1^2 \quad \forall v \in V. \tag{2.1}$$

In this chapter, we will examine the case $d = 2$. Let $v \in U$ be arbitrary and suppose that a solution $u \in U$ of (1.1) exists. Multiplying (1.1) by the function $v - u \in V$ and then integrating over Ω, we get by Green's theorem the following variational equality

$$a(u, v - u) + \int_{\Gamma_2} \beta u^4 (v - u) ds = F(v - u) \quad \forall v \in U.$$

From here we obviously get the variational inequality

$$a(u, v - u) + \int_{\Gamma_2} \beta u^4 (v - u) ds \geq F(v - u) \quad \forall v \in U. \tag{2.2}$$