Journal of Computational Mathematics, Vol.16, No.1, 1998, 63–80.

SOME ESTIMATES WITH NONCONFORMING ELEMENTS IN DOMAIN DECOMPOSITION ANALYSIS*[1]

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Abstract

Some essential estimates, especially the so–called extension theorems, are established in this paper, for the nonconforming finite elements with their continuity at the vertices or the edge midpoints of the elements of the quasi–uniform mesh. As in the conforming discrete cases, these estimates play key roles in the theoretical analysis of the nonoverlap domain decomposition methods for the solving of second order self–adjoint elliptic problems discretized by the nonconforming finite element methods.

Key words: Nonconforming finite element, extension theorem, domain decomposition, elliptic problem.

1. Introduction

For simplicity of the exposition, we consider the elliptic boundary value problem on a bounded open polygonal domain $\Omega \subset \Re^2$

$$u \in H^1(\Omega) : \begin{cases} a(u,v) = (f,v), & \forall\, v \in H_0^1(\Omega) \\ u = g, & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where

$$a(u,v) = \int_\Omega \nabla u \nabla v, \quad (f,v) = \int_\Omega fv, \quad f \in H^{-1}(\Omega), \quad g \in H^{\frac{1}{2}}(\partial\Omega).$$

It is well–known that (1.1) has a unique solution $u \in H^1(\Omega)$ (cf.[7, 15, 16]).

Suppose that $\Omega_h = \{e\}$ is a quasi–uniform mesh of Ω, i.e., Ω_h satisfies

$$\sup_{e\in\Omega_h} \inf_{B_r \supset e} r \le ch, \qquad \inf_{e\in\Omega_h} \sup_{B_r \subset e} r \ge Ch, \tag{1.2}$$

where e, a triangle or a quadrilater, represents the typical element in Ω_h, B_r is a region bounded by the circle of radius r, $h = \max_{e\in\Omega_h} h_e$ is the mesh parameter and

* Received June 23, 1995.
[1)] This work was supported by the National Natural Science Foundation of China and the Tsinghua University Fund for Science.

$h_e = \inf_{B_r \supset e} r$. Here and later, c and C, with or without subscript, denote generic positive constants independent of h. Let V_h be the finite element space on Ω_h and π_h be the corresponding interpolation operator. V_h can be the space of Wilson elements[5], Carey membrane elements[4] or Wilson–like elements[14], which are continuous at the vertices of each $e \in \Omega_h$. Also, V_h can be the space of Crouzeix–Raviart elements[6] or quartic rectangular elements[13], which are continuous at the edge midpoints of each $e \in \Omega_h$. For briefness, the former is called the nonconforming elements of the first kind and the latter is called the nonconforming elements of the second kind. V_h can be written in the following general form

$$\begin{aligned} V_h = \{v : v|_e \ &\text{is a polynomial of finite order, } v \text{ is continuous at the} \\ &\text{vertices (edge midpoints) of } e, \quad \forall\, e \in \Omega_h\}, \\ V_h^0 = \{v \in V_h :\ &v(x) = 0,\ \forall \text{ interpolation point } x \in \partial\Omega\}. \end{aligned}$$

Denote $A(w,v) = \sum_{e\in\Omega_h} \int_e \nabla w \nabla v$, $|v|_{1,\Omega,h} = \sqrt{A(v,v)}$. Obviously, $A(\cdot,\cdot), |\cdot|_{1,\Omega,h}$ are the inner product of V_h^0 and its induced norm respectively. The nonconforming finite element discrete problem of (1.1) is

$$u_h \in V_h : \begin{cases} A(u_h, v) = (f, v), & \forall\, v \in V_h^0 \\ u_h(x) = g(x), & \forall \text{ interpolation point } x \in \partial\Omega \end{cases} \tag{1.3}$$

With the development of parallel computers, domain decomposition methods have recently become an important focus in the field of computational mathematics. By now, all kinds of domain decomposition algorithms have been developed to solve the algebraic system of equations arising from the discretization of (1.1) via the conforming finite element methods. It is noted that several fundamental inequalities, especially the so–called extension theorems play key roles in the theoretical analysis of those nonoverlap domain decomposition algorithms (substructuring methods)[2,3,20]. Therefore, when considering the nonconforming finite element discrete problem (1.3), we should establish those inequalities in V_h correspondingly. For this purpose, the conforming interpolation operator I_h is introduced to act as a bridge between V_h and the piecewise linear continuous finite element space where many inequalities have already been constructed[2,5,18]. Since the regularity of the solution u of (1.1) depends on the domain Ω (cf.[7, 15, 16]), we investigate advanced error estimations of the nonconforming approximate solution u_h of (1.1) under weaker assumption on the regularity. In this way, we eventually establish a series of essential estimates in V_h, some of which are the extension theorems[8,10], the Poincaré inequalities and the maximum norm estimate.

The remainder of this paper is organized as follows: Sect.2 gives advanced error estimations of (1.3). Sect.3 introduces the conforming interpolation operator I_h and analyses its properties. Sect.4 describes and proves some essential estimates in V_h to conclude the paper.

For the length of the present paper, we omit here their applications to the theoretical analysis of nonoverlap domain decomposition methods for the solving of (1.3), which can be referred to [8, 9, 11, 12].