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H-SPLITTINGS AND ASYNCHRONOUS PARALLEL ITERATIVE METHODS*

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Abstract

The paper discusses H-splitting and H-compatible splitting furthermore, some properties are given. Asynchronous parallel multisplitting algorithm and its generalization for linear systems $Ax = b$ are established. Convergence of these algorithms is proved under given conditions. The convergent range of relaxation factor ω is given, numerical example is shown.

1. Properties of H-splitting

A. Frommer and D.B. Szyld[3] proposed H-splitting and H-compatible splitting for two-stage methods. But they didn't discuss two splitting furthermore. We will show some properties of H-splitting and H-compatible splitting before we apply them to establish asynchronous parallel multisplitting algorithm.

Definition 1. ([3]) *Given $A \in L(R^n)$, $A = M - N(M,\ N \in L(R^n))$, which is called H-splitting, if $\langle M\rangle - |N|$ is an M-matrix; which is called H-compatible splitting, if $\langle A\rangle = \langle M\rangle - |N|$. Where $\langle A\rangle$ is Ostrowski matrix, $|N|$ is absolution matrix.*

Obviously, for an H-matrix, an H-compatible splitting is an H-splitting, but an H-splitting is not necessarily an H-compatible splitting. For example:

$$A = \begin{bmatrix} 1 & 0.25 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} - \begin{bmatrix} 0 & -0.25 \\ -1 & 0 \end{bmatrix} = M - N$$
$$\langle M\rangle - |N| = \begin{bmatrix} 1 & -0.25 \\ -3 & 1 \end{bmatrix} \neq \langle A\rangle.$$

But $\langle M\rangle - |N|$ is an M-matrix.

Property 1. *Given $A \in L(R^n)$, let $A = M - N$ be an H-splitting. Then A is an H-matrix.*

Proof. By definition, $\langle M\rangle - |N|$ is an M-matrix.

$$\langle A\rangle = \langle M - N\rangle \geq \langle M\rangle - |N|$$

By comparison property of M-matrix, $\langle A\rangle$ is an M-matrix.

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Hence, A is an H-matrix.

Property 2. *Given $A \in L(R^n)$, let $A = M - N$ be an H-splitting. Then $\rho(\langle M\rangle^{-1}|N|) < 1$ and $\rho(|M^{-1}N|) < 1$.*

By definition, $\langle M\rangle - |N|$ is an M-matrix, hence $\langle M\rangle$ is an M-matrix. This implies that $\langle M\rangle - |N|$ is a convergent regular splitting, then $\rho(\langle M\rangle^{-1}|N|) < 1$.

By $\langle M\rangle^{-1} \geq |M^{-1}|$, we have

$$|M^{-1}N| \leq |M^{-1}||N| \leq \langle M\rangle^{-1}|N|.$$

Hence, $\rho(|M^{-1}N|) < 1$.

Property 3. *Let A be a nonsingular H-matrix, let $A = M_i - N_i (i = 1, 2)$ be H-compatible splittings. If $|N_1 \geq |N_2|$, then $\rho(\langle M_1\rangle^{-1}|N_1|) \geq \rho(\langle M_2\rangle^{-1}|N_2|)$.*

Proof. By definition of H-compatible splitting and comparison property of regular splittings [1], conclusion is proved directly.

But it is not true for H-splitting, for example:

$$A = \begin{bmatrix} 1 & -0.25 \\ -1 & 1 \end{bmatrix} = M_1 - N_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0.25 \\ 1 & 0 \end{bmatrix},$$

$$A = \begin{bmatrix} 1 & -0.25 \\ -1 & 1 \end{bmatrix} = M_2 - N_2 = \begin{bmatrix} 1 & 0 \\ -1.5 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0.25 \\ -0.5 & 0 \end{bmatrix},$$

$$|N_1| \geq |N_2|, \quad \rho(\langle M_2\rangle^{-1}|N_2|) = \frac{3 + \sqrt{41}}{16} > 0.56, \quad \rho(\langle M_1\rangle^{-1}|N_1|) = 0.5.$$

2. Asynchronous Parallel Algorithm

The parallel multisplitting iterative method for solving large systems of linear algebraic equations

$$Ax = b, \quad A \in L(R^n) \quad x, b \in R^n \tag{2.1}$$

was first presented by O'Leary and White[4] in 1985. Since then, many papers have dealt with parallel multisplitting iterative methods for linear and nonlinear problems[4]–[10] etc. In order to make use of parallel computer efficiently, a great deal of research is currently being focused on asynchronous implementation, in which computation and communication are performed independently in each processor so that processor idle time is reduced, time of convergence is shorted and so on.

In this section, we will establish asynchronous parallel iterative methods based on H-splitting.

Let $A = M_i - N_i (i = 1, 2, \cdots, l)$ be H-splittings, $E_i (i = 1, 2, \cdots, l)$ are nonnegative diagonal matrices and $\sum_{i=1}^{l} E_i = I$, where I is the identity matrix. (2.1) changes into following equivalent form:

$$x = \sum_{i=1}^{l} E_i M_i^{-1} N_i x + \sum_{i=1}^{l} E_i M_i^{-1} b, \tag{2.2}$$