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RQI DYNAMICS FOR NON-NORMAL MATRICES WITH REAL EIGENVALUES*1)

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Abstract

RQI is an approach for eigenvectors of matrices. In 1974, B.N Parlett proved that it was a "succeessful algorithm" with cubic convergent speed for normal matrices. After then, several authors developed relevant theory and put this research into dynamical frame. [3] indicated that RQI failed for non-normal matrices with complex eigenvalues.

In this paper, RQI fornon-normal matrices with only real spectrum is analyzed. The authers proved that eigenvectors are super-attractive fixed points of RQI. The geometrical and topological behaviours of two periodic orbits are considered in detail.

The existness of three or higher periodic orbits and their geometry are considered in detail.

The existness of three or higher periodic orbits and their geometry are still open and of interest. It will be reported in our forthcomming paper.

1. Introduction

As well known, RQI (Rayleigh Quotient Iteration) is a practical algorithm for eigenvalue problems of symmetric matrices. In 1974, B.N. Parlett proved that the sequence generated by RQI always converges to an eigenvector for almost all of initial vectors if the matrix in question is a normal one. Namely the set of vectors in R^n, for which RQI diverges, has zero measure. Nevertheless, he also pointed out the convergent speed being cubic[1]. In 1989, S. Barttson and J. Smillie considered RQI for symmetric matrix again. They discovered that the dynamics of RQI is, in a sense, similar to that of Morse-Smale diffeomorphism except its discontinuity. In their paper[2] the geometry and topology of initial vectors for which RQI is covergent are characterized.

Based upon discrete dynamical system and bifurcation theory, S. Barttson and J. Smillie, in 1990, constructed an example to show the existness of a nonempty open set of matrices for which RQI strongly fails. This set is refered to be a bad set. Note that the example given by the authers has eigenvalues with nonzero imaginary parts. Comparing with the counterpart of Newton iteration for polynomail equations[4,5], the authers of [3] gave rise to an open question:

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Is successful RQI for matrices having only real spectrum?

In this paper, the authers solved this problem partially. In section 2, RQI is overviewed briefly. Then the relationship between it and discrete dynamical system is described. Section 3 is devoted to our new results. Finally, a conjecture is presented.

2. R.Q.I. Algorithm and Discrete Dynamical Systems

R.Q.I. Algorithm is a will-known method for symmetric eigenproblems. In fact, it is nothing but the inverse power iteration with shifts. We summarize RQI briefly as follows:

Let A be a $n \times n$ real matrix, $\rho(x)$ be Rayleigh Quotient defined on $R^n \setminus \{0\}$ as

$$\rho A(x) = \frac{(x, Ax)}{(x, x)}$$

where $(\cdot, \cdot)$ is Euclid inner product.

Algorithm 2.1. (*R.Q.I. Algorithm*)

Step 1. Choose an initial vector $x_0 in R^n \setminus \{0\}$.

Step 2. For $k = 0, 1, 2, \cdots$,

if $(A - \rho(x_k)I)$ is singular

then get an eigenvector and normalize it, Stop

else

$$Y_{k+1} = (A - \rho A(x_k)I)^{-1}(x_k) \equiv F_A(x_k)$$

Step 3. Normalize y_{k+1} and x_{k+1}

Step 4. Go to Step 2.

First of all, well define a discrete dynamical system for RQI. Note that a nonzero vector is an eigenvector of matrix A. if and only if each element in its one-dimensional span is also an eiginvector. The set $\{\alpha x | \alpha \in R\}$ forms a one-dimensional subspace in R^n. All such subspaces compose a manifold of $n-1$ dimension, RP^{N-1}, refered as a projective space. One can view the projective space as providing a space of eigenvector candidates.

It is easy to verify, $\alpha \neq 0$,

$$\begin{aligned} \rho A(\alpha x) =& \rho A(x) \\ F_A(\alpha x) \equiv (A - \rho A(\alpha x)I)^{-1}(\alpha x) =& \alpha(A - \rho A(x)I)^{-1}x = \alpha F_A(x). \end{aligned}$$

RQI defines a smooth map F_A on the subset of RP^{n-1} for which ρA does not yield an eigenvalue of A. If $\rho A(x)$. In the event that $\rho A(x)$ is a repeated eigenvalue then the dimension of the eigenspace is greater than 1. To have a well-defined iteration we must specify a method for the selection of the particular eigenvector. For dynamical reasons we define $F_A(x)$ to be the one-dimensional subspace spanned by the orthogonal projection of x onto the eigenspace corresponding to $\rho A(x)$. If x is orthogonal to the eigenspace the choice of eiggenvector is dynamically unimportant and we can specify any algorithm for choosing the eigenvector. Of course, the fiscrete dynamical system may be possibly discontinuous with respect to x.