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MUSCL TYPE SCHEMES AND DISCRETE ENTROPY CONDITIONS*1)

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Abstract

In this paper, the semi-discrete entropy conditions with so called the proper discrete entropy flux of a class of high resolution MUSCL type schemes are discussed for genuinely nonlinear scalar conservation laws. It is shown that the high resolution schemes satisfying such semi-discrete entropy conditions cannot preserve second order accuracy in the rarefaction region.

1. Introduction

Consider the hyperbolic conservation laws:

$$\frac{\partial u}{\partial t}+\frac{\partial f(u)}{\partial x}=0$$
$$u(x,0)=u_0(x). \tag{1.1}$$

The research of numerical methods for equations (1.1) has been developed rapidly in this decade. Since the appearance of the concept of TVD(total variation diminishing) schemes, various high resolution schemes have been proposed[1,2,3,4] and successfully applied to computational fluid dynamics. It is well known that the convergence of the numerical methods for hyperbolic conservation laws depends on the entropy condition of the numerical solutions[5]. Previously the construction of difference schemes was always based on some kinds of total variation stability (TVD, TVB, and ENO etc.). In order to satisfy the entropy condition the constructed schemes are modified in such a way by introducing some quantities depending on the grid width[6]. Generally, the difference schemes for homogenious problems like (1.1) only depend on the grid ratio but independ on the grid width itself explicitly. So, it is meaningful to construct schemes satisfying the entropy condition without introducing quantities depending explicitly on the grid width. Merriam[7] and Sonar[8] put out the concept of the proper discrete entropy flux,

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that is, discretizating the entropy flux in such a proper way that the entropy condition can be satisfied and simultaneously the difference solution satisfies some kind of total variation stability. In [9], Zhao and Wu discussed the relationship between entropy conditions and high resolution schemes for 1-D scalar linear conservation laws, and obtained second order accurate TVD schemes using limiters. In [10], Zhao and Tang discuss the relationship between the discrete entropy conditions. The MmB property for linear scalar hyperbolic conservation laws in two dimensions are presented in [11].

In this paper, we discuss the accuracy of high resolution MUSCL type schemes and their semi-discrete entropy condition for 1-D genuinely nonlinear conservation laws. In section 2, we define the concept of the proper discrete entropy flux and discuss the entropy condition of three point monotone schemes. In section 3, the results of section 2 are generalized to five point MUSCL type schemes. The relationship between proper discrete entropy conditions and the TVD property of high resolution MUSCL type schemes is analyzed. Our main result is that the high resolution schemes of MUSCL type satisfying semi-discrete entropy conditions with proper discrete entropy flux cannot preserve second order accuracy in the region of rarefaction.

2. Three Point Monotone Schemes and Entropy

To begin with, let us consider the three point monotone semi-discrete schemes in conservative form

$$\frac{\partial}{\partial t}u_j + \frac{1}{\Delta x}(h_{j+\frac{1}{2}} - h_{j-\frac{1}{2}}) = 0 \tag{2.1}$$

$$h_{j+\frac{1}{2}} = h(u_{j+1}, u_j) \quad , \quad h(u,u) = f(u) \tag{2.2}$$

where Δx is the variable meshsize in x-direction. The schemes (2.1), (2.2) is monotone if

$$\frac{\partial h(v,w)}{\partial v} \le 0 \quad \text{and} \quad \frac{\partial h(v,w)}{\partial w} \ge 0 \; . \tag{2.3}$$

As is well known that the weak solution of (1.1) is not unique. Let $U(u)$ be any convex function, the so-called entropy function, and corresponding function $F(u)$, the entropy flux satisfying $F'(u) = U'(u)f'(u)$. (U, F) is called an entropy pair. If the weak solution u of (1.1) satisfies the inequality:

$$\frac{\partial U(u)}{\partial t} + \frac{\partial F(u)}{\partial x} \le 0 \tag{2.4}$$

in the sense of distribution for every entropy pair (U, F), then u is the unique physical solution of (1.1). The inequality (2.4) is called the entropy inequality (or the entropy condition).

Corresponding to the conservative scheme (2.1), the semi-discrete entropy inequality is defined as

$$\frac{\partial}{\partial t}U(u_j) + \frac{1}{\Delta x}(H_{j+\frac{1}{2}} - H_{j-\frac{1}{2}}) \le 0 \tag{2.5}$$