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L^∞ CONVERGENCE OF TRUNC ELEMENT FOR THE BIHARMONIC EQUATION[*1)]

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Abstract

The paper considers the L^∞ convergence for TRUNC finite elements solving the boundary value problems of the biharmonic equation. The nearly optimal L^∞ estimates for the error of first order derivatives are given.

The TRUNC plate element is proposed and developed by Argyris et al.. The numerical experiences show that the element has very good results[1,2]. The mathematical proof of convergence of the TRUNC element is given by Shi Zong-ci in paper [3]. This paper will consider the L^∞ convergence for the TRUNC plate element.

1. The TRUNC Plate Element

Given a triangle T with vertices $a_i = (x_i, y_i), i = 1, 2, 3$, we denote by λ_i the area coordinates for the triangle T and put

$$\begin{aligned} &\xi_1 = x_2 - x_3, \quad \xi_2 = x_3 - x_1, \quad \xi_3 = x_1 - x_2, \\ &\eta_1 = y_2 - y_3, \quad \eta_2 = y_3 - y_1, \quad \eta_3 = y_1 - y_2. \end{aligned}$$

The nodal parameters of the element are the function values and the values of the two first derivatives at the vertices of the triangle T. According to paper [3], on the triangle T the shape function is an incomplete cubic polynomial,

$$\begin{aligned} w = {}& b_1\lambda_1 + b_2\lambda_2 + b_3\lambda_3 + b_4\lambda_1\lambda_2 + b_5\lambda_2\lambda_3 + b_6\lambda_3\lambda_1 \\ &+ b_7(\lambda_1^2\lambda_2 - \lambda_1\lambda_2^2) + b_8(\lambda_2^2\lambda_3 - \lambda_2\lambda_3^2) + b_9(\lambda_3^2\lambda_1 - \lambda_3\lambda_1^2), \end{aligned} \tag{1.1}$$

which is uniquely determined by the nine nodal parameters $w_i, w_x(i), w_y(i), i = 1, 2, 3$. The coefficients b_i are determined as follows,

$$\begin{cases} b_i = w_i, \quad i = 1, 2, 3, \\ b_4 = -\frac{1}{2}\{(w_x(1) - w_x(2))\xi_3 + (w_y(1) - w_y(2))\eta_3\}, \\ b_5 = -\frac{1}{2}\{(w_x(2) - w_x(3))\xi_1 + (w_y(2) - w_y(3))\eta_1\}, \\ b_6 = -\frac{1}{2}\{(w_x(3) - w_x(1))\xi_2 + (w_y(3) - w_y(1))\eta_2\}, \\ b_7 = w_1 - w_2 - \frac{1}{2}(w_x(1) + w_x(2))\xi_3 - \frac{1}{2}(w_y(1) + w_y(2))\eta_3, \\ b_8 = w_2 - w_3 - \frac{1}{2}(w_x(2) + w_x(3))\xi_1 - \frac{1}{2}(w_y(2) + w_y(3))\eta_1, \\ b_9 = w_3 - w_1 - \frac{1}{2}(w_x(3) + w_x(1))\xi_2 - \frac{1}{2}(w_y(3) + w_y(1))\eta_2. \end{cases} \tag{1.2}$$

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The shape form (1.1) with (1.2) is another one of Zienkiewicz's element. This element is a C^0 element, nonconforming for plate bending problems, which converges to the true solution only for very special meshes. The TRUNC element is obtained by modifying the variational formulation.

Let Ω be a convex polygon in $R^2, f \in L^2(\Omega)$. Consider the plate bending problem with the clamped boundary conditions,

$$\begin{cases} \triangle^2 u = f, & \text{in } \Omega \\ u|_{\partial\Omega} = \dfrac{\partial u}{\partial N}\Big|_{\partial\Omega} = 0 \end{cases} \tag{1.3}$$

The weak form of the problem (1.3) is to find $u \in H_0^2(\Omega)$ such that,

$$a(u,v) = (f,v), \qquad \forall v \in H_0^2(\Omega), \tag{1.4}$$

where

$$a(u,v) = \int_\Omega (\triangle u \triangle v + (1-\sigma)(2u_{xy}v_{xy} - u_{xx}v_{yy} - u_{yy}v_{xx}))\, dxdy,$$
$$(f,v) = \int_\Omega fv\, dxdy, \tag{1.5}$$

and $0 < \sigma < \frac{1}{2}$ is the Poisson ratio.

Dividing Ω into a regular family T_h of triangular elements T with diameters $h_T \le h$, and defining on each triangle T the shape function in the form (1.1) and (1.2), we obtain the finite element space V_h. Then, the standard finite element approximation of problem (1.4) is to find $u_h \in V_h$ such that

$$a_h(u_h, v_h) = (f, v_h), \qquad \forall v_h \in V_h, \tag{1.5}$$

where

$$a_h(u,v) = \sum_T \int_T (\triangle u \triangle v + (1-\sigma)(2u_{xy}v_{xy} - u_{xx}v_{yy} - u_{yy}v_{xx}))\, dxdy. \tag{1.6}$$

The modification of variational formulation (1.6) is carried out as follows. Every function $v_h \in V_h$ can be splited into two parts,

$$v_h = \bar{v}_h + v_h', \tag{1.7}$$

where

$$\bar{v}_h|_T = a_1\lambda_1 + a_2\lambda_2 + a_3\lambda_3 + a_4\lambda_1\lambda_2 + a_5\lambda_2\lambda_3 + a_6\lambda_3\lambda_1, \tag{1.8}$$

representing a full quadratic polynomial on T, and

$$v_h' = a_7(\lambda_1^2\lambda_2 - \lambda_1\lambda_2^2) + a_8(\lambda_2^2\lambda_3 - \lambda_2\lambda_3^2) + a_9(\lambda_3^2\lambda_1 - \lambda_3\lambda_1^2), \tag{1.9}$$

being a cubic polynomial. Define a new discrete bilinear form,

$$b_h(v_h, w_h) = a_h(\bar{v}_h, \bar{w}_h) + a_h(v_h', w_h'), \quad \forall v_h, w_h \in V_h. \tag{1.10}$$