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DISCRETE APPROXIMATIONS FOR SINGULARLY PERTURBED BOUNDARY VALUE PROBLEMS WITH PARABOLIC LAYERS, III[*1)]

P.A. Farrell
(*Department of Mathematics and Computer Science, Kent State University, USA*)

P.W. Hemker
(*CWI, Center for Mathematics and Computer Science, Amsterdam, The Netherlands*)

G.I. Shishkin
(*IMM, Institute of Mathematics and Mechanics, Ural Branch of the Russian Academy of Science, Ekaterinburg, Russia*)

Abstract

In this series of three papers we study singularly perturbed (SP) boundary value problems for equations of elliptic and parabolic type. For small values of the perturbation parameter parabolic boundary and interior layers appear in these problems. If classical discretisation methods are used, the solution of the finite difference scheme and the approximation of the diffusive flux do not converge uniformly with respect to this parameter. Using the method of special, adapted grids, we can construct difference schemes that allow approximation of the solution and the normalised diffusive flux uniformly with respect to the small parameter.

We also consider singularly perturbed boundary value problems for convection-diffusion equations. Also for these problems we construct special finite difference schemes, the solution of which converges ε-uniformly. We study what problems appear, when classical schemes are used for the approximation of the spatial derivatives. We compare the results with those obtained by the adapted approach. Results of numerical experiments are discussed.

In the three papers we first give an introduction on the general problem, and then we consider respectively (i) Problems for SP parabolic equations, for which the solution and the normalised diffusive fluxes are required; (ii) Problems for SP elliptic equations with boundary conditions of Dirichlet, Neumann and Robin type; (iii) Problems for SP parabolic equation with discontinuous boundary conditions.

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Part III

PARABOLIC EQUATIONS WITH A DISCONTINUOUS BOUNDARY CONDITION

1. Introduction

The solution of partial differential equations that are singularly perturbed and/or have discontinuous boundary conditions generally have only limited smoothness. Due to this fact difficulties appear when we solve these problems by numerical methods. For example for regular parabolic equations with discontinuous boundary conditions, classical methods (FDM or FEM) on regular rectangular grids do not converge in the ℓ^∞-norm on a domain that includes a neighbourhood of the discontinuity [8, 9, 4].

If the parameter multiplying the highest-order derivative vanishes, boundary- and interior layers generally appear. When a discontinuity is present in the initial function (given at $t = 0$), an interior layer is generated. Outside a neighbourhood of the discontinuity classical difference schemes converge in the ℓ^∞-norm for each fixed value of the small parameter, but they do not converge in the ℓ^∞-norm in the neighbourhood of the discontinuity. Neither do they converge uniformly in ε in any neighbourhood of the interior layer [8, 9]. Therefore, it is interesting to construct special methods which are ℓ^∞-convergent for parabolic PDEs with discontinuous initial functions, both in the regular and in the singularly perturbed case. In the latter case it is important to see if and when such convergence can be uniform in the small parameter on the whole domain of definition.

In [8, 9] singularly perturbed parabolic equations with discontinuous boundary conditions were studied. There, special difference schemes were constructed for these problems. In order to be able to construct a method that was uniformly convergent (in the small parameter ε), special variables were used in the neighbourhood of the discontinuity. By introducing the variables $\theta = x/(2\varepsilon\sqrt{t})$ and t, the singularity was removed from the boundary value problem and the solution became a smooth function in the new variables. This behaviour of the transformed solution allows the use of a classical scheme in the transformed variables in the neighbourhood of the singularity. Away from the singularity the classical scheme can be used with the original variables.

This transformation in the neighbourhood of the singularity implied the use of a specially condensed grid in the neighbourhood of the boundary and interior layers. So we can say that the technique is based on: (i) fitted methods in which the coefficients of the difference equations are adapted to the singularities; (ii) methods that use special, refined meshes in the neighbourhood of singularities. For these schemes ℓ^∞-convergence on the whole domain is proved, uniformly in the small parameter, but a disadvantage of these schemes is that they are very hard to realise in practice.