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ON THE SPLITTINGS FOR RECTANGULAR SYSTEMS*

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Abstract

Recently , M. Hanke and M. Neumann[4] have derived a necessary and sufficient condition on a splitting of $A = U - V$, which leads to a fixed point system , such that the iterative sequence converges to the least squares solution of minimum 2-norm of the system $Ax = b$. In this paper, we give a necessary and sufficient condition on the splitting such that the iterative sequence converges to the weighted Moore-Penrose solution of the system $Ax = b$ for every $x_0 \in C^n$ and every $b \in C^m$. We also provide a necessary and sufficient condition such that the iterative sequence is convergent for every $x_0 \in C^n$.

1. Introduction

It is well-known that the most prevalent approach for obtaining a fixed point system of the following system

$$Ax = b, \qquad A \in C^{m\times n} \tag{1.1}$$

is via a splitting of the coefficient matrix A into

$$A = U - V. \tag{1.2}$$

If $m = n$ and U is nonsingular, we present the equivalent formulation of (1.1) by

$$x = U^{-1}Vx + U^{-1}b. \tag{1.3}$$

If $m \neq n$ or if U is not invertible, we can, by taking a generalized inverse U^- of U (instead of U^{-1}), extend (1.3) by considering the fixed point system

$$x = U^-Vx + U^-b. \tag{1.4}$$

Generalized inverses of matrices play a key role in our present work. It is instructive for our purposes to think of reflexive inverses as weighted Moore-Penrose inverses and to call the corresponding solution which induce weighted Moore-Penrose solution. In section 2, we summarize preliminary results from the literature on generalized inverses

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which are most relevant to this paper briefly. In section 3 , we derive a necessary and sufficient condition for a splitting (1.2) to yield a fixed point iterative scheme such that the limit point $\bar{x}$ is a weighted Moore-Penrose solution to (1.1). In section 4, we provide a necessary and sufficient condition such that the iterative sequence is convergent for every $x_0 \in C^n$ and every $b \in C^m$. In section 5, a numerical experiment is presented to illustrate the performance of the splitting.

2. Preliminary and Background Results

Let $A \in C^{m\times n}$ and suppose $X \in C^{n\times m}$. Then X is called a reflexive inverse of A if

$$AXA = A \quad \text{and} \quad XAX = X. \tag{2.1}$$

Given a subspace $R \subseteq C^n$ which is complementary to $N(A)$ and a subspace $N \subseteq C^m$ which is complementary to $R(A)$, then there exists a unique reflexive inverse X of A such that

$$R(X) = R \quad \text{and} \quad N(X) = N \tag{2.2}$$

and conversely, if X is a reflexive inverse of A, then $R(X)$ and $N(X)$ are complementary subspace of $N(A)$ and $R(A)$. In the following we shall use R(A) and N(A) to denote the range and the nullspace of a matrix A. Accordingly we write $A^-_{R,N} := X$. It is known that

$$A^-_{R,N}A = P_{R,N(A)} \quad \text{and} \quad AA^-_{R,N}, = P_{R(A),N}, \tag{2.3}$$

where $P_{R,N(A)}$ and $P_{R(A),N}$ denote the projectors on R along $N(A)$ and on $R(A)$ along N, respectively.

With any reflexive inverse X of A one can associate two vector norms, one in $C^n$and one in C^m, as follows:

$$\|x\|_{R,N(A)} := (\|P_{R,N(A)}x\|_2^2 + \|(I - P_{R,N(A)})x\|_2^2)^{1/2}, \forall x \in C^n$$

and

$$\|y\|_{R(A),N} := (\|P_{R(A),N}y\|_2^2 + \|(I - P_{R(A),N})y\|_2^2)^{1/2}, \forall y \in C^m.$$

Due to the finite dimensional setting which we work in, for any vector $b \in C^m$ the set

$$\delta_b := \{\bar{x} \in C^n : \|b - A\bar{x}\|_{R(A),N} = \inf_{x\in C^n} \|b - Ax\|_{R(A),N}\} \neq \phi \tag{2.4}$$

and the vector $\bar{z} := A^-_{R,N}b$ has the following properties:

$$\bar{z} \in \delta_b \text{ and } \|\bar{z}\|_{R,N(A)} = \min_{\bar{x}\in\delta_b} \|\bar{x}\|_{R,N(A)}. \tag{2.5}$$

Therefore we can interpret any reflexive inverse as a weighted Moore-Penrose inverse and vice versa $\bar{z}$ as a weighted Moore-Penrose solution to the system $Ax = b$.

We next mention some choices of R and N which correspond to reflexive inverses that are frequently used in applications and in the literature. First,suppose that $N =$