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THE CONVERGENCE OF MULTIGRID METHODS FOR SOLVING FINITE ELEMENT EQUATIONS IN THE PRESENCE OF SINGULARITIES*1)

Y.Q. Huang Y.X. Li
(*Department of Mathematics, Xiangtan University, Hunan, China*)

Abstract

We analyze the convergence of multigrid methods applied to finite element equations of second order with singularities caused by reentrant angles and abrupt changes in the boundary conditions. Provided much more weaker demand of classical multigrid proofs, it is shown in this paper that, for symmetric and positive definite problems in the presence of singularities, multigrid algorithms with even one smoothing step converge at a rate which is independent of the number of levels or unknowns. Furthermore, we extend this result to the nonsymmetric and indefinite problems.

1. Introduction

Multigrid Methods provide optimal order solvers for linear systems of finite element equations arising from elliptic boundary value problems. The convergence of multigrid methods was proved by many authors[2−6,9−12]. All these proofs, require strong regularities and quasi-uniformity of grids[3,10]. For example, assuming $H^{1+\alpha}$ regularity and quasi-uniform triangulations, Bank & Dupont[3] showed a convergence rate of $O(m^{\frac{-\alpha}{2}})$, for a growing number m of smoothing steps per level. In the optimal case $\alpha = 1$, the problem has to be H^2–regular. When the region has reentrant angles or abrupt changes in the boundary condition, H^2–regularity is violated, and in addition, the approximation properties of the finite element space deteriorate because of the presence of singularities not captured by the quasi-uniform grids.

Yserentant[11] proved the convergence of multigrid methods for symmetric and definite problems with singularities. However, a sufficiently large number of smoothing steps m was required. Shangyou Zhang[12] got the similar result using nonnested multigrid methods, but it also assumed that m is larger than a certain constant. In this

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work, we prove the convergence of multigrid methods for symmetric and definite problems with singularities of even one smoothing step. Furthermore, it is shown in this paper that, multigrid methods applied to indefinite and nonsymmetric problems also converge on nonquasiuniform grids.

The outline of the remainder of the paper is as follows.

In section 2, we define a weighted function $\phi_r(x)$ and a family of triangulations governed by $\phi_r(x)$, and describe a j-level multigrid iterative procedure. An important lemma is given in section 3. In section 4, we prove our multigrid convergence theorems. We provide some results for nonsymmetric and indefinite problems with singularities in section 5.

Throughout this paper, c and C will denote generic positive constants which may take on different values in different places. These constants will always be independent of the mesh parameters.

2. Notation and Multigrid Scheme

For simplicity, we consider the model problem

$$\begin{aligned} -\Delta u + u &= f, &&\text{in } \Omega,\\ u &= 0, &&\text{on } \Gamma_D,\\ \frac{\partial u}{\partial n} &= 0, &&\text{on } \Gamma_N, \end{aligned} \tag{1}$$

where Ω is an open bounded polygonal domain in R^2 with the boundary subdivided into two parts Γ_D and Γ_N. Let x_i, $1 \le i \le M$, denote the vertices of Ω with θ_i, where θ_i is the interior angle of Ω at x_i. Because of possible changes in the boundary conditions, the case $\theta_i = \pi$ is permitted. Let $0 < \theta_i < 2\pi$. For each vertex x_i, we define $k_i = 1$ if the two sides of x_i belong either both to Γ_D or both to Γ_N, and $k_i = 1/2$ otherwise. Let $\alpha_i = \min(1, (k_i\pi)/\theta_i)$, then $1/4 \le \alpha_i \le 1$ holds. If we have pure Dirichlet or Neumann boundary conditions, $\alpha_i < 1$ only holds for reentrant angles. We choose r_i with $1 - \alpha_i \le r_i < 1$ if $\alpha_i < 1$, and $r_i = 0$ if $\alpha_i = 1$. Define

$$\phi_r(x) = \prod_{i=1}^{M} \mid x - x_i \mid^{r_i} \tag{2}$$

for $r = (r_1, r_2, \ldots, r_M)$, where $\mid x \mid$ denotes the Euclidean norm. We assume that the family $T_0, T_i \ldots$of triangulations has the following two properties[1]: Let $\tau \in T_j$ be a triangle, then

$$ch_j\phi_r(x) \le d(\tau) \le ch_j\phi_r(x), \quad \text{if } \phi_r(x) \ne 0, \quad \forall x \in \tau, \tag{3}$$

$$ch_j \max_{x\in\tau} \phi_r(x) \le d(\tau) \le ch_j \max_{x\in\tau} \phi_r(x), \quad \text{if } \phi_r(x) \ne 0 \quad \text{for some } x \in \tau. \tag{4}$$