Journal of Computational Mathematics, Vol.13, No.4, 1995, 306–314.

HERMITE-TYPE METHOD FOR VOLTERRA INTEGRAL EQUATION WITH CERTAIN WEAKLY SINGULAR KERNEL*1)

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Abstract

We discuss the Hermite-type collocation method for the solution of Volterra integral equation with weakly singular kernel. The constructed approximation is a cubic spline in the continuity class C^1. We prove that this method is convergent with order of four.

1. Introduction

This paper considers the numerical solution of the second-kind Volterra integral equation

$$y(t) + (Ky)(t) = g(t), \tag{1.1}$$

where $y(t)$ is the unknown solution, $g(t)$ is a given function and K is the integral operator for some given kernel function K,

$$(Ky)(t) = \int_0^t K(\frac{t}{s})y(s)\frac{1}{s}ds. \tag{1.2}$$

Such equations arise from certain diffusion problems. Because K is not compact, so the standard stability proofs for numerical methods do not fit.

Many people have worked on Hermite-type collocation methods for second-kind Volterra integral equations with smooth kernels[3,4,5,6], but very few deal with weakly singular kernels. Papatheodorou & Jesanis (1980) considered Volterra integro-differential equations with weakly singular kernels. Diogo, Mckee & Tang (1991) investigated a Hermite-type collocation method for (1.1) with a singular kernel of the form $K(\sigma) = \frac{1}{\sqrt{\pi}\sqrt{\ell_n \sigma}\sigma^\mu}$, $\mu > 1$. They also considered two low-order product integration methods for the solution of (1.1) with a singular kernel of the form $K(\sigma) = \frac{1}{\sqrt{\pi}\sqrt{\ell_n \sigma}\sigma^\mu}$[10]. For general kernel $K(\sigma)$, no papers have appeared to discuss it.

* Received March 26, 1993.
1) The Project was Supported by National Natural Science Foundation of China.

In this paper, first we would to show that a unique smooth solution exists when $\alpha = \int_1^\infty \frac{|K(\sigma)|}{\sigma} d\sigma < 1$. The basic idea is to derive two (linear) Volterra equations for $y(t)$ and $y'(t)$ by transforming the original integral equation. Having the coupled equations for both $y(t)$ and $y'(t)$, we can then employ piecewise cubic Hermite polynomials to obtain numerical solution of (1.1). Finally, the convergence analysis is given.

2. Preliminaries

Let $C^m[0.T]$ denote the Banach space of mth order derivative continuous real-valued functions with the uniform norm

$$|| \ u \ ||_{m,\infty} = \max_{0 \le j \le m} \max_{0 \le t \le T} |u^{(j)}(t)|.$$

Our assumption on K is

$$\alpha = \int_1^\infty \frac{| \ K(\sigma) \ |}{\sigma} d\sigma < 1. \tag{2.1}$$

Lemma 1. *If $g \in C^m[0,T]$ and (2.1) is satisfied, then (1.1) possesses a unique solution $y \in C^m[0,T]$.*

Proof: Choosing an arbitrary function $v(t) \in C^m[0,T]$, and defining $u = S(v)$ such that

$$u(t) + \int_0^t K(\frac{t}{s}) v(s) \frac{1}{s} ds = g(t), \quad t \in [0,T] \tag{2.2}$$

where $S(v) = -\int_0^t K(\frac{t}{s}) v(s) \frac{1}{s} ds + g(t)$.
Setting $s = \lambda t$ we have

$$\int_0^t K(\frac{t}{s}) v(s) \frac{1}{s} ds = \int_0^1 K(\frac{1}{\lambda}) v(\lambda t) \frac{1}{\lambda} d\lambda \ . \tag{2.3}$$

Since $v \in C^m[0,T]$ and $g \in C^m[0,T]$, we obtain from (2.2) and (2.3) that

$$u^{(j)}(t) = -\int_0^1 K(\frac{1}{\lambda}) v^{(j)}(\lambda t) \lambda^{j-1} d\lambda + g^{(j)}(t), \tag{2.4}$$

where $0 \le j \le m$. If $u_1 = S(v_1)$ and $u_2 = S(v_2)$, we have

$$\begin{aligned} |u_1^{(j)} - u_2^{(j)}| \ & \le \int_0^1 |K(\tfrac{1}{\lambda})| \lambda^{j-1} |v_1^{(j)}(\lambda t) - v_2^{(j)}(\lambda t)| d\lambda \\ & \le \int_0^1 |K(\tfrac{1}{\lambda})| \lambda^{-1} d\lambda \cdot || \ v_1 - v_2 \ ||_{m,\infty} \ . \end{aligned} \tag{2.5}$$

Noting that the coefficient of the last term of (2.5) equals α, it follows that

$$|| \ u_1 - u_2 \ ||_{m,\infty} \le \alpha \ || \ v_1 - v_2 \ ||_{m,\infty} \ . \tag{2.6}$$

The inequality (2.6) implies that the operator S is a contraction mapping. Since C^m is a complete normed space, S has a unique fixed point $y(t) \in C^m[0,T]$ such that $y = S(y)$. This completes the proof.