Journal of Computational Mathematics, Vol.13, No.2, 1995, 123-129.

A FINITE DIFFERENCE METHOD FOR THE MODEL OF WHEEZES*)

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Abstract

In this paper, a finite difference scheme for the linear and nonlinear models of wheezes are given. The stability of the finite difference scheme for the linear model is obtained by using of von Neumann method. Moreover, the convergence and stability of the finite difference scheme for the nonlinear model are studied by the energy inequalities method. By some numerical computations, the relationships between angular frequency and wall position, fluid speed and amplitude are discussed. Finally, the author shows that the numerical results are coincided with Grotberg's theoretical results.

1. Introduction

In order to study the pitch of wheezes in patients, J.B.Grotberg and others have given a class of mathematical model of wheezes[1,2]:

$$
\begin{cases}
u = \Phi_x, \qquad w = \Phi_z, & (1.1)\\
\triangle\Phi = 0, & (1.2)\\
\Phi_t + \frac{1}{2}\nabla\Phi\cdot\nabla\Phi + 2R_f\Phi + P - P_a = 0, & (1.3)\\
MW_{tt} + 2R_wW_t + BW_{xxxx} + 1 + W + \beta(1+W)^3 - TW_{xx} + P - P_e = 0. & (1.4)
\end{cases}
$$

Where $\triangle$ and ∇ are the Laplace operator and gradient operator, respectively. The Cartesian components (u, w) are the dimensionless axial fluid velocity and dimensionless vertical fluid velocity respectively. $\Phi(x, z, t)$ is the velocity potential function, P is the dimensionless fluid pressure determined from the unsteady Bernoulli equation (1.3), P_a is the steady driving pressure, P_e is the external pressure. M, R_w, B, β and T are wall-to-fluid mass ratio, dimensionless wall damping coefficient, bending stiffness to elastance ratio, nonlinear elastance coefficient and applied longitudinal tension to elastance ratio, respectively. The geometry and physical parameters of the problem are indicated in Fig.1.

Fig. 1

* Received October 15, 1991.

Generally, $\Phi(x,z,t)$ is a travelling wave like in [1,2]:

$$\Phi(x,z,t) = Ai\frac{\omega - kS}{k\sinh k}\cosh(kz)\mathrm{e}^{\mathrm{i}\theta} \tag{2}$$

where ω is the dimensionless angular frequency, k is the wave number defined by $k = 2\pi b/L$ and L is the dimensionless wavelength, S is the dimensionless fluid speed, A is an arbitrary constant, $\theta = kx - \omega t$. It is very easy to check that (2) satisfies (1.2). Let $P_e = P_a - 2R_f Sx - S^2/2$[1]. In this paper, we shall discuss the periodic problem with the travelling wave $\Phi(x,z,t)$, then equation (1) can be rewritten as the following:

$$\begin{cases} MW_{tt} + 2R_w W_t + BW_{xxxx} + q(1+W) - TW_{xx} = f(x,t), & (x,t)\epsilon R\times I, \quad (3.1)\\ W(x+\lambda,t) = W(x,t), & (x,t)\epsilon R\times I, \quad (3.2)\\ W(x,0) = W_0(0), & x\epsilon R, \quad (3.3)\\ W_t(x,0) = W_t(x), & x\epsilon R. \quad (3.4)\end{cases}$$

Where $f(x,t) = [\Phi_t + \frac{1}{2}\nabla\Phi\cdot\nabla\Phi + 2R_f\Phi]\ |_{z=-1}\ -(2R_f Sx + S^2/2)$ and $q(1+W) = 1 + W + \beta(1+W)^3$. W is an unknown function. R is the real line and $I = [0, T_1]$.

J.B.Grotberg and other authors[1,2] discussed the standing wave solutions for system (1). While the solutions are not standing wave, the methods in [1,2] are invalid. Zho Yulin[3] gave the weighted difference schemes, but the weighted coefficient $\alpha islimitedin[\frac{1}{2}, 1]$. Therefore the difference schemes are implicit. In this paper, we shall give an explicit difference approximation and study the convergence and stability.

2. Finite Difference Scheme and It's Stability for the Linear Equation

Let τ, h be time-step and space-step lengths, respectively. $J = [\lambda/h]$, $N = [T_1/\tau]$. $x_j = jh$, $t_n = n\tau$. $0 \le j \le J$, $0 \le n \le N$. Where $[y]$ denotes the larger integer which is not greater than y. Notation W_j^n is the approximation of $W(x_j,\ t_n)$. A finite difference scheme is given as follows:

For $0 \le j \le J$

$$\begin{cases} MW^n_{jt\bar{t}} + 2R_w W^n_{j\hat{t}} + BW^n_{jxx\bar{x}\bar{x}} + q(1+W^n_j) - TW^n_{jx\bar{x}} = f^n_j, & 0 < n < N, \quad (4.1)\\ W^n_{j+rj} = W^n_j, & 0 \le n \le N, \quad (4.2)\\ W^0_j = W_0(x_j), & (4.3)\\ W^0_t = W_t(x_j). & (4.4)\end{cases}$$

Where $W^n_{jt} = (W^{n+1}_j - W^n_j)/\tau$, $W^n_{j\bar{t}} = (W^n_j - W^{n-1}_j)/\tau$ and $W^n_{j\hat{t}} = (W^{n+1}_j - W^{n-1}_j)/(2\tau)$. Similarly, we can define W^n_{jx} and $W^n_{j\bar{x}}$.

In the following we consider the von Neumann stability of the finite difference scheme for the linear equation of (3), *i.e.* $q(1+W) = 1+W$, $\beta = 0$. Say $e^n_j = W^n_j - \hat{W}^n_j$, where W^n_j and $\hat{W}^n_j$ are the solution of the finite difference scheme for the